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PROCEEDINGS
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NATIONAL ACADEMY OF SCIENCES

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GALACTIC AND EXTRAGALACTIC STUDIES, XXI. DISTRIBUTION OF 95,000 GALAXIES IN THE NORTH GALACTIC CAP

By Harlow Shapley

Harvard College Observatory

Communicated March 1, 1951

Among the regions in which surveys of galaxies have been made in the course of a census of the inner Metagalaxy is the important North Galactic Polar Cap, covering the 2760 square degrees from $\beta = +60°$ to $\beta = +90°$. The distribution in this area, heretofore unpublished, is now summarized for galaxies brighter than photographic magnitude 17.5. The deeper surveys, now in progress with Schmidt cameras, should soon provide sampling counts to at least the nineteenth magnitude.

1. The galaxy counts summarized in table 1 were made on plates of two series, all of three-hour exposures. The 57 A plates, made with the Bruce telescope in South Africa, show stars to magnitude 18.2 on the average; the star limit for the 69 MC plates, made at the Oak Ridge Station with the Metcalf doublet, averages 17.9. In declination the Polar Cap extends from $-2°$ to $+58°$. North of declination $+22°$ the MC instrument was used and the area south of $+22°$ (about 40 per cent of the Polar Cap), except for MC 33360 at $+19°.4$, was covered with the southern telescope.

Earlier investigations with these instruments show that the galaxy "completeness" limit m_n is 0.4 mag. brighter than the star limit m_s for the A plates,[1] and 0.57 mag. brighter for MC plates.[2] The difference $m_s - m_n$ varies from plate to plate because of slightly varying conditions in seeing and focus. The survey therefore lacks uniformity because of this spread and also because of the residual uncertainties in the magnitude sequences which are based as usual on the Seares and van Rhijn star-count tables.

2. In order to obtain a uniform picture of the density distribution of galaxies to magnitude 17.5, it has been necessary to reduce the galaxy counts on each plate by the application of the factor $f = 10^{0.6(18.07 - m_s)}$

TABLE 1. GALAXIES IN NORTH GALACTIC CAP

PLATE NUMBER	GALACTIC LONG.	GALACTIC LAT.	N_9	m_s	$\bar{N}_T$	PLATE NUMBER	GALACTIC LONG.	GALACTIC LAT.	N_9	m_s	$\bar{N}_T$
A6718	245°	+73°	285	17.8	36.4	MC27704	98°	+74°	206	18.1	22.0
A6719	258	+80	366	17.8	46.8	MC28719	106	+57	319	18.1	34.0
A6720	262	+75	204	17.9	22.7	MC28754	115	+67	339	17.8	64.3
A13810	255	+70	521	17.9	57.9	MC28788	129	+68	216	18.1	23.0
A14551	232	+77	423	17.9	47.0	MC28868	120	+58	185	18.3	15.0
A14699	257	+64	173	17.9	19.2	MC28914	63	+59	194	17.8	31.2
A15251	251	+64	95	17.1	31.9	MC28938	100	+69	279	18.0	34.1
A15363	280	+80	88	17.1	29.6	MC29265	142	+58	420	18.4	29.4
A15370	278	+74	125	17.6	21.0	MC29283	137	+64	273	18.5	16.7
A15384	296	+73	158	17.7	23.3	MC29316	198	+80	568	18.1	60.6
A15389	249	+69	278	18.1	23.4	MC29377	70	+68	161	17.8	25.9
A16086	263	+60	411	17.5	79.5	MC29382	185	+69	279	18.1	29.8
A16090	344	+62	241	17.9	26.8	MC29447	58	+64	179	17.9	25.1
A16407	341	+69	347	18.0	33.6	MC29494	40	+70	357	17.7	66.2
A16757	324	+59	231	17.8	29.6	MC29498	73	+57	283	17.8	45.6
A17345	267	+65	726	18.0	70.3	MC29526	303	+85	283	18.0	34.6
A17468	194	+66	329	17.5	63.6	MC29530	354	+69	437	17.9	61.2
A17721	335	+74	500	17.9	55.6	MC29532	341	+80	454	18.1	48.4
A17761	298	+70	1168	18.2	85.7	MC29606	52	+59	163	17.8	26.2
A17774	328	+60	362	18.3	23.1	MC29612	32	+65	304	17.3	98.0
A18291	313	+60	604	18.4	33.6	MC29945	149	+61	288	18.3	23.3
A18296	311	+68	503	18.5	24.5	MC29949	138	+71	251	18.0	30.7
A18391	255	+59	441	18.2	32.3	MC29962	47	+65	233	18.3	18.9
A18489	189	+62	312	18.0	30.2	MC29978	146	+66	306	18.0	37.4
A18506	336	+64	538	18.6	22.7	MC29979	41	+61	142	18.1	15.1
A19063	212	+60	424	17.9	47.2	MC29981	27	+74	464	17.8	74.8
A19349	218	+78	862	17.8	110.1	MC29986	124	+76	390	17.8	62.9
A19422	320	+71	670	18.4	37.2	MC29987	31	+59	268	17.7	49.8
A19522	286	+72	876	18.0	84.8	MC29988	158	+62	186	17.8	30.0
A19986	294	+65	668	18.3	42.7	MC29989	22	+68	198	17.5	48.4
A19990	321	+78	540	18.0	52.2	MC30038	74	+74	447	18.0	54.7
A20095	304	+63	618	18.2	45.4	MC30041	83	+59	210	17.7	39.0
A20107	210	+67	628	18.1	53.0	MC30045	170	+62	170	17.8	27.3
A20115	351	+62	387	17.6	64.9	MC30129	55	+70	329	18.1	35.1
A20117	220	+71	673	17.6	112.9	MC30734	73	+79	288	18.0	35.2
A20670	281	+66	657	18.4	36.4	MC30740	131	+58	612	18.1	65.3
A20705	226	+60	548	18.3	35.0	MC30797	97	+64	366	17.8	59.0
A20862	198	+70	918	18.4	51.0	MC30837	72	+63	169	17.6	35.9
A20880	219	+64	582	18.1	49.2	MC30851	85	+64	394	17.8	63.4
A20890	269	+70	1635	18.6	69.0	MC30879	102	+79	145	18.0	17.8
A20918	276	+65	1092	18.8	34.9	MC31292	125	+63	192	17.6	40.8
A20922	268	+60	796	18.6	33.6	MC31358	230	+85	124	17.7	23.0
A21719	241	+62	658	18.3	42.0	MC31359	85	+69	212	17.8	34.1
A21926	320	+64	1766	18.6	74.5	MC31365	158	+86	344	18.0	42.0
A22393	296	+81	1435	18.5	69.7	MC31368	60	+83	124	17.7	23.0
A22464	203	+63	882	18.3	56.2	MC31431	166	+81	299	17.5	73.1
A22569	227	+67	834	18.7	30.6	MC31432	142	+79	360	17.7	66.8
A22575	206	+74	944	18.4	52.4	MC31440	10	+80	176	17.9	24.6
A23089	296	+63	278	18.4	15.4	MC31442	35	+79	413	17.7	76.7
A23153	280	+64	418	18.4	23.2	MC31444	50	+75	214	17.5	52.3
A24821	306	+74	240	18.2	17.6	MC32130	154	+74	411	17.6	87.2
A24823	327	+68	861	18.0	83.0	MC32148	174	+76	196	17.5	47.9
A26301	285	+62	488	18.5	23.7	MC32157	159	+66	224	17.6	47.6
A26304	332	+57	548	18.5	26.7	MC32226	358	+65	471	17.9	65.9
A26704	230	+74	1712	18.8	54.8	MC32281	9	+68	173	17.6	36.7
A26848	340	+58	622	18.4	34.6	MC32284	191	+75	277	17.4	77.6
A26857	238	+70	430	18.0	41.6	MC32867	172	+68	564	17.8	90.9
MC27104	105	+65	336	18.4	23.6	MC32872	180	+63	497	17.7	92.2
MC27134	16	+87	838	18.5	51.2	MC32908	350	+75	462	17.8	74.4
MC27599	111	+84	277	18.1	29.6	MC32914	10	+63	363	17.9	50.8
MC27639	115	+72	371	18.3	30.1	MC33347	7	+75	542	18.0	66.2
MC27641	108	+62	614	18.5	37.6	MC33360	244	+81	166	18.0	20.3
MC27688	95	+59	262	17.9	36.7	MC33924	20	+62	250	17.4	70.0

for the MC plates, and $f = 10^{0.6(17.9 - m_s)}$ for the A plates, where m_s, in the fifth column of table 1, is the observed stellar magnitude limit.

It follows that $N_{9r} = fN_9$, where N_9 is the total number of galaxies found on each plate in a central area 3° on a side, and N_{9r} is the number we would expect to find if the respective plates gave a complete census of galaxies to magnitude 17.5, and no fainter. Applying these reductions to N_9 in the fourth column of table 1, and dividing by 9, we get the number of galaxies per square degree ($\bar{N}_r$ in the last column), reduced to magnitude 17.5.

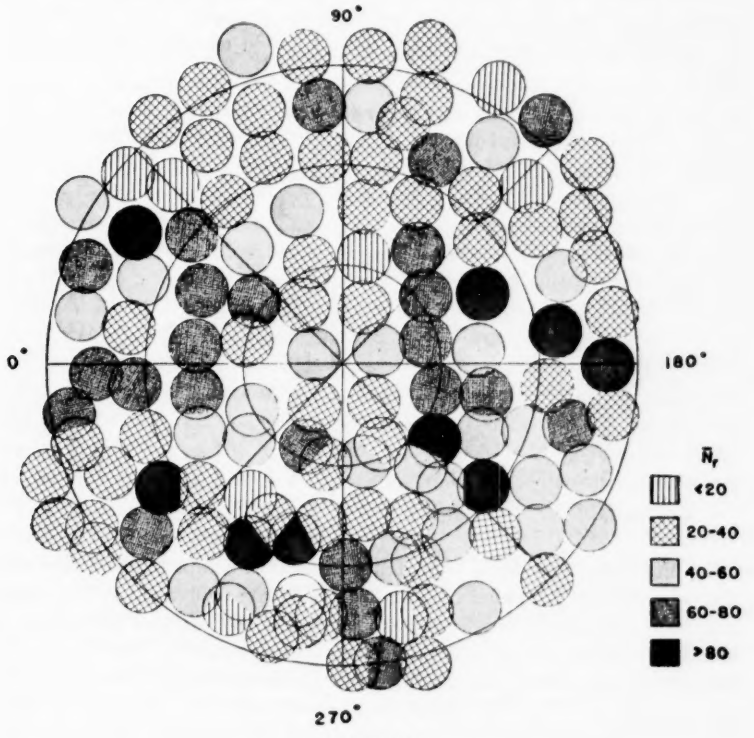

FIGURE 1

For example, when an A plate shows stars to magnitude 18.3, and 362 galaxies are counted in the central 9 square degrees, we find

$$N_{9r} = 362 \cdot 10^{0.6(17.9 - 18.3)} = 208,$$

and $\bar{N}_r = 23.1$ is the computed average number of galaxies per square degree for that plate.

In making this reduction we have assumed, as heretofore, that we have, on the average, a uniform radial frequency of galaxies. There is, as figure 1 shows, marked unevenness in the frequency from plate to plate. Frequently there is irregular distribution on the individual plates. The method is therefore dependable only for the mean of many plates. It is the best we can do in the absence of magnitudes for the individual galaxies.

On 11 MC plates the magnitudes of galaxies 17.5 and brighter have been directly determined in earlier studies.[3] The difference $\Delta\bar{N}$ between the measured galaxies per square degree for these MC plates and the computed numbers of table 1 for the same plates, averages 3 per cent, or less than one galaxy per square degree, the computed numbers being slightly larger.

TABLE 2

REGIONAL DISTRIBUTION

REGION	TOTAL N_{17}	NUMBER OF PLATES	$\bar{N}_{17}$	$\bar{m}_1$
$\lambda 0°$ to $\lambda 90°$	11,468	28	45.5	14.73
$\lambda 90$ to $\lambda 180$	11,214	30	41.5	14.80
$\lambda 180$ to $\lambda 270$	15,324	34	50.1	14.66
$\lambda 270$ to $\lambda 360$	13,140	34	42.9	14.77
$\lambda 0$ to $\lambda 180$	22,682	58	43.5	14.76
$\lambda 180$ to $\lambda 360$	28,464	68	46.5	14.71
$\lambda 90$ to $\lambda 270$	26,538	64	46.1	14.72
$\lambda 270$ to $\lambda 90$	24,608	62	44.1	14.75
$\beta +57$ to $\beta +70$	27,855	75	41.3	14.80
$\beta +70$ to $\beta +80$	18,103	37	54.4	14.60
$\beta +80$ to $\beta +90$	5,188	14	41.2	14.80
All plates	51,146	126	45.1	14.74

3. For figure 1 we have indicated the essential completeness of the survey over the cap by taking the area for each plate equivalent to 25 square degrees. The plates actually cover nearly 35 square degrees, but for the A plates the quality for nebular discovery and measurement falls off outside the central 9 square degrees, and for the MC plates outside the central 25 square degrees. The MC plates require essentially no correction in their central 25 square degrees.[2] If we had used these larger MC areas, the values of $\bar{N}_r$ would not have been changed appreciably except for the plates:

		25 sq. deg.		9 sq. deg.
MC27134	where $\bar{N}_r$ would be	33.6	instead of	51.2
31368		29.8		23.0
31440		62.0		24.6
32148		68.5		47.9
32872		34.1		92.2

In the mean, for all 69 MC plates, we have

$$\frac{\bar{N}_{r(9)} - \bar{N}_{r(25)}}{\bar{N}_{r(9)}} = -4.2.$$

From this small percentage difference we conclude again that distance correction on MC plates can be ignored in a galaxy census, and that the total number of galaxies entering the survey of the Polar Cap can appropriately be increased from the 55,686 of column 4, table 1, to 94,880. In addition, the outer unused parts of the A plates show approximately 35,000 galaxies, mostly brighter than magnitude 17.0.

4. Figure 1 shows the irregularities in surface distribution that are now generally recognized as characteristic of the inner Metagalaxy. When extensive areas are considered the unevenness is largely smoothed out. Table 2 illustrates the comparability of the four quadrants of galactic longitude, and the still closer similarity when the Cap is divided into halves. When the means are taken for intervals of galactic latitude, the greatest richness is found between latitudes $+70°$ and $+80°$, as figure 1 also indicates.

Scattered over the Polar Cap are several clusters of galaxies, mostly with their individual members fainter than magnitude 17.5. A few of these were known before the present survey. No attention is paid to them at this time because the Schmidt cameras can soon outline and analyze them far better after adding two magnitudes to the depth of the survey.

Finally it should be noted that this North Galactic Cap is much richer to magnitude 17.5 than we have found (Harvard, Mount Wilson, Lick surveys) for the high-latitude sky at large. In the formula for uniform radial space density, $\log N = 0.6(m - m_1)$, the space-density parameter, m_1, for the whole sky falls between 15.1 and 15.2. (This is the magnitude to which the average survey must reach in order to find one galaxy per square degree.) The last column of table 2 shows that in the North Galactic Cap m_1 is 14.74, that is, four-tenths of a magnitude brighter, corresponding to 1.75 times the average density. In the South Galactic Polar Cap we found some years ago[1] $m_1 = 15.16 \pm 0.02$ (m.e.) from a study of some 13,000 galaxies. This important inequality in registered population between north and south apparently cannot be attributed to differences in space absorption near the galactic plane (with the south having much heavier absorption). When galaxies to magnitude 19.0 are examined, the inequality between the north and the south has largely disappeared, and with it the hypothesis that local space absorption is responsible. The inequality was pointed out earlier[4] as one of the indications of the existence of significant radial and transverse density gradients in the distribution of faint galaxies.

Of the several assistants who have worked on the survey of the North Galactic Cap, Miss Ann Hearn has contributed most in bringing the investi-

gation to a conclusion. The MC plates were all made by Mr. H. A. Sawyer, and the Bruce plates under the direction of Dr. J. S. Paraskevopoulos.

Summary.—Working with long-exposure plates made with the Bruce and Metcalf doublets at the South African and Oak Ridge Stations of the Harvard Observatory, we have investigated, in the course of the survey of the inner Metagalaxy, the distribution of approximately 95,000 galaxies brighter than magnitude 17.5 in the North Galactic Polar Cap.

About 95 per cent of the area with galactic latitudes greater than +60° is covered by 126 plates, but in only 75 per cent of the area have the galaxies been tabulated, since the counts could not be used dependably for the outer portions of the plates.

The average population of galaxies over the whole Cap, 45.1 per square degree, to magnitude 17.5 (which corresponds to a limiting distance in the polar caps of about 80 million light years for the galaxy of average luminosity) appears to be about 1.75 times that of the South Galactic Cap. The distribution diagram and tables 1 and 2 present fully the survey and the irregularities.

The important Virgo and Coma clusters of galaxies lie within the area surveyed, but the latter is too faint and the former too sparse to make significant contributions to the census.

[1] These PROCEEDINGS, **26,** 169 (1940), Harvard Reprint 194.

[2] *Harvard Annals,* **106,** 4 (1938); these PROCEEDINGS, **26,** 560 (1940), Harvard Reprint 208. A mean value has been used in this paper.

[3] Seyfert, C. K., *Harvard Annals,* **105,** No. 10 (1937); these PROCEEDINGS, **26,** 554–561 (1940), Harvard Reprint 208.

[4] *American Scientist,* **32,** 67 (1944), Harvard Reprint 255.

CURVE OF GROWTH POLARIZATION

BY JAMES W. WARWICK

WHITIN OBSERVATORY, WELLESLEY COLLEGE[*]

Communicated by D. Menzel, February 23, 1951

Introduction.—When gas is submerged in a magnetic field, the atomic absorption lines split into components, the amount of the splitting being proportional to the strength of the magnetic field. The conditions of polarization of the absorbed light vary according to the angle between the magnetic field and the direction of propagation of the light. For each component, one can imagine the incident light split into two orthogonal states of polarization, one of which cannot be absorbed by the component.

Magnetic fields of the strength observed in sunspots and stars are frequently strong enough to separate completely the components of normal

triplets lying on the 45° or flat parts of the curve of growth. Each component then acts independently of the others on its own two states of polarization, passing one completely, and absorbing only the other.

In case of complete separation, one can form the curve of growth separately for each component, if he knows the profile of the component as a function of frequency and effective numbers of atoms. By means of a suitable polarizing analyzer one can eliminate, for any given direction of observation, the state of polarization which is passed by a given component. One can then determine the profile of the component in units of the continuum of polarized light in the state of polarization which can be absorbed by the component.

We shall assume that the equivalent width of the component, measured in this way in a polarized continuum, depends on the absorption coefficient in the component in a simple, idealized manner. The absorption coefficient, in turn, depends on the angle made by the line of sight with the magnetic field. For all angles such that the optical depth at component center, X, is less than a certain value, X_0, we assume that the equivalent width, W, varies proportionally to X. When X is greater than X_0, we assume that W is constant, and equals W_0 (Fig. 1).

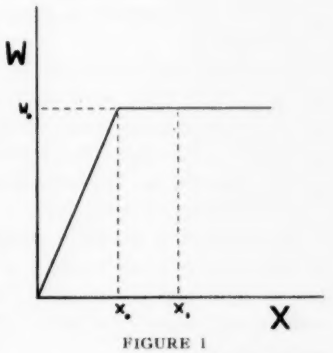

FIGURE 1

Assumed curve of growth.

Babcock[1] used this idealized curve of growth in his analysis of the magnetic intensification of stellar absorption lines. The justification for this procedure is that the ordinary curve of growth[2] for unpolarized lines varies linearly with optical depth for small enough numbers of absorbing atoms, and comes to a point of inflection for moderate numbers of absorbing atoms. The slope at the point of inflection is small.

It is not necessarily true that the equivalent width of a component, considered as a function of its directionally dependent absorption coefficient, will vary in the same way as an unpolarized line, considered as a function of optical depth. However, in either case, the errors made in assuming a simplified form of the curve of growth are probably greater than the error of considering polarized components and unpolarized lines to have curves of growth of the same shape.

We shall assume conditions of saturation such that the σ components of a triplet lie, for all directions, on the flat part of the curve of growth. This means (Fig. 1) that each σ component has an optical depth greater than X_0,

for any direction of observation. On the other hand, the single π component of the triplet does not absorb at all in the longitudinal effect. The π component reaches the point X_0 for some direction (determined by the saturation of the σ components) intermediate between the longitudinal and transverse directions. From that direction up to the transverse direction its optical depth increases, but its equivalent width remains constant at W_0.

In the case of the triplet, there are two σ components seen transversally and one π component. All three components have the same equivalent width, W_0, in our special conditions of saturation. Each σ component absorbs light from the polarized continuum characterized by an electric vector vibrating in the direction perpendicular to the magnetic field. The π component, on the other hand, absorbs from the orthogonal continuum, characterized by an electric vector vibrating in the direction parallel to the magnetic field.

If we block out the continuum parallel to the magnetic field by means of a Nicol prism, we may expect to observe an equivalent width $2W_0$. If we block the continuum perpendicular to the field, we may observe equivalent width W_0. The equivalent width of the line, as measured through a Nicol prism, depends on the orientation of the analyzer. This effect is called "curve of growth polarization."

Computation of Stellar Equivalent Widths.—The optical depth in either σ component is given by $X_1 (1 + \cos^2 \gamma)$, where γ is the angle between the line of sight and the magnetic field. X_1 is the optical depth in either σ component, when $\gamma = \pi/2$. The assumed conditions of saturation are such that $X_1 \geq X_0$.

If the conditions of saturation had been chosen so that the π and both σ components had optical depths less than X_0, for the transverse case, then no curve of growth polarization would have been present. The optical depth in the π component varies as $\sin^2 \gamma$. For the present case of saturated components we therefore write the optical depth in the π component as $2X_1 \sin^2 \gamma$.

The optical depth in the π component is X_0 when $\sin^2 \gamma = X_0/2X_1$. The choice of a value of X_0/X_1 determines the saturation conditions. For reasons of convenience, we take $X_0/X_1 = 1/2$. Then, for $\sin \gamma < 1/2$, the π component lies on the part of the curve of growth for which W increases proportionally to X.

On the proportional part of the curve of growth, $W/W_0 = X/X_0$. If we define the unit of equivalent width to be W_0, and the unit of optical depth, X_0, we have $W = X$. The unit of equivalent width is measured in the continuum of polarized light which can be absorbed by the component. Therefore, the equivalent widths in figure 1 must be divided by 2 to refer to the total, unpolarized continuum.

The angle δ is defined to be the angle made between the projection of the

magnetic field on the plane of the sky, and the direction of the magnetic dipole axis, which we shall assume lies in the plane of the sky. Following Babcock, we write the stellar magnetic field, as a function of position on the surface of the star, in the form

$$\begin{aligned} H_x &= (3/2)\, H_p \sin\theta \cos\theta \cos\varphi \\ H_y &= (3/2)\, H_p \sin\theta \cos\theta \sin\varphi \\ H_z &= H_p(\cos^2\theta - 1/2 \sin^2\theta). \end{aligned} \tag{1}$$

$H_{x,y,z}$ are the Cartesian components of the magnetic field; H_z is along the dipole axis and H_x along the line of sight. θ and φ are spherical coordinates defined in the usual way with respect to the Cartesian coordinates. We introduce the parameters

$$\alpha = \frac{10 + 6\cos 2\theta}{(3 \sin 2\theta)^2} \tag{2}$$

and

$$\beta = \frac{(1 + 3\cos 2\theta)^2}{(3 \sin 2\theta)^2}, \tag{3}$$

in terms of which one can verify that

$$\cos^2\gamma = \frac{\cos^2\varphi}{\alpha} \tag{4}$$

and

$$\tan^2\delta = \frac{\sin^2\varphi}{\beta}. \tag{5}$$

Also,

$$\alpha - \beta = 1. \tag{6}$$

We have

$$\sin^2\gamma \cos^2\delta = \frac{\beta}{\alpha}. \tag{7}$$

The stellar radiation field is assumed to be of the form

$$I = c(A + B\mu), \tag{8}$$

where we set the radiation from the center of the disk equal to c by taking

$$A + B = 1. \tag{9}$$

We normalize I so that the total flux from the visible hemisphere equals unity. Thus, we determine c so that

$$2\pi c \int_0^1 (A + B\mu)\mu \, d\mu = 1,$$

or

$$c = \frac{3}{\pi} \frac{1}{(3 - B)}. \tag{10}$$

One may write the equivalent widths of the components, in terms of the angles γ and δ, in the forms presented in table 1. The entries of table 1 are the equivalent widths of a single component as measured in the unpolarized continuum in the state of polarization parallel or perpendicular to the magnetic dipole axis.

TABLE 1

LOCAL EQUIVALENT WIDTHS OF COMPONENTS

	PARALLEL	PERPENDICULAR	γ
π component	$\frac{1}{2}\cos^2\delta$	$\frac{1}{2}\sin^2\delta$	$\gamma > \pi/6$
π component	$2\sin^2\gamma\cos^2\delta$	$2\sin^2\gamma\sin^2\delta$	$\gamma < \pi/6$
σ component	$\frac{1}{2}\frac{\cos^2\gamma\cos^2\delta + \sin^2\delta}{1+\cos^2\gamma}$	$\frac{1}{2}\frac{\cos^2\gamma\sin^2\delta + \cos^2\delta}{1+\cos^2\gamma}$	all γ

The longitudinal effect of the σ components represents the equivalent width the line would have had had there been no Zeeman splitting at all, since the two σ components absorb in orthogonal states of polarization in the longitudinal effect. Because of the flux normalization of the radiation field, the integrated equivalent width over the whole stellar disk would have been unity, if there had been no Zeeman splitting. The appropriately weighted sums of the equivalent widths in table 1 will therefore represent the intensification factor introduced by Babcock.

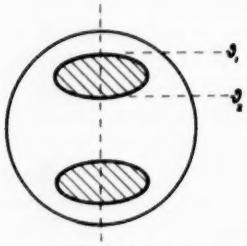

FIGURE 2

The projected stellar disk. Regions for which $\gamma < \pi/6$ are hatched.

Figure 2 represents the projected stellar disk. Within the hatched areas, $\gamma < \pi/6$. For θ lying in the interval θ_1 to θ_2, the integration of the expressions of table 1 is in two parts, corresponding to $\gamma \gtrless \pi/6$.

There are two practical ways to find the numerical values of the integrals arising from table 1: (1) cubature, or (2) analytic integration over φ, followed by quadrature in θ. I have followed the latter course. The analytic integrals are all elementary, but somewhat tedious and lengthy, and will not be reproduced here. The functions resulting from these integrations were evaluated for twelve equally spaced intervals in each of the ranges 0° to θ_1, θ_1 to θ_2 and θ_2 to 90°. The θ corresponding to the maximum lateral extent of the cross-hatched area represents a latitude of arc $\sin\sqrt{3}/3 = 35^\circ.3$. This is the latitude at which the meridional magnetic field is pointed at the observer. The projected height of the zone between θ_1 and θ_2 is 0.416 radii, and its projected width, 1.000 radii.

The equivalent widths were obtained from Weddle's six-interval rule

of quadratures. The formulae expressing the equivalent widths measured in the perpendicular and parallel states were found to be

$$W_\perp = \frac{1}{(3 - B)} (2.19 - 0.65B) \tag{11}$$

and

$$W_\parallel = \frac{1}{(3 - B)} (2.07 - 0.71B). \tag{12}$$

Typical values of B are used to form table 2, whose last column contains essentially the polarization.

Summary and Conclusions.—It appears that 4 or 5 per cent plane polarization is quite likely to be present in the absorption lines of the magnetic stars for some directions of observation. This prediction is based on: (1) an idealized curve of growth theory; (2) well-separated Zeeman components; (3) the stellar magnetic field oriented like that of a dipole; (4) the star a sphere; (5) darkening to the limb following the cosine law; (6) the root-mean-square atomic velocity being constant over the surface of the star; (7) the magnetic dipole axis being in the plane of the sky; and (8) the line being a Zeeman triplet.

TABLE 2

POLARIZATION OF A TRIPLET IN A MAGNETIC STAR

B	$W_\perp + W_\parallel$	$W_\perp - W_\parallel$
0.00	1.42	0.04
0.50	1.43	0.06
1.00	1.45	0.09

Whether such a small amount of polarization is actually observable is difficult to say. Struve, in attempting to observe resonance polarization in the ionized calcium emission lines of AR Lacertae, estimated that with his (photographic) apparatus, only 8 per cent polarization would be the minimum detectable.[3] It is possible that photoelectric methods may be able to reduce this value considerably—although the general problem of observing the equivalent width of a stellar absorption line photoelectrically has not yet been solved in practice.

If a reliable and small lower limit to the actual polarization can be set from direct observation, then it would appear that certain detailed conclusions about the geometry of the star would be possible. From the symmetry of the magnetic field, it follows that a star viewed along the dipole axis ought to show no polarization. Intermediary directions will show polarization up to the equatorial maximum of 4 or 5 per cent. The direction of the polarization will always be such that the maximum equivalent width will be observed at right angles to the direction of the projected dipole axis, whether the star is viewed equatorially or not.

If the not-unreasonable assumption is made that a star's magnetic dipole

and rotation axes are coincident, then we have here at least a theoretical basis for a determination of the direction of the projected rotation axis on the plane of the sky.

* NOTE: The work leading up to the writing of this paper was done at the Harvard College Observatory during the spring of 1950.

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[3] Struve, O., *Ibid.*, **108**, 155 (1948).

HORMONAL INFLUENCE ON NUCLEAR SYNTHESIS. I. ESTROGEN AND UTERINE GLAND NUCLEI

BY MAX ALFERT AND HOWARD A. BERN

DEPARTMENT OF ZOOLOGY, UNIVERSITY OF CALIFORNIA, BERKELEY

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In a recent publication[1] Salvatore has pointed out that injection of estrogen into ovariectomized rats results in a doubling of nuclear volume in uterine gland cells. Progesterone does not produce this effect. Consequently the author concluded that his observations and other evidence "clearly show(s) the genic nature of rhythmical growth" and that "estrogen acts on the genes by inducing their reproduction."

Inasmuch as these conclusions were based solely on karyometric analysis, it seemed desirable to test Salvatore's hypothesis of genic duplication by comparing the amounts of some nuclear components in uterine gland nuclei before and after doubling of nuclear volume. Recently developed photometric techniques[2, 3] make it possible to measure amounts of protein as well as desoxyribonucleic acid (DNA) in single nuclei. Furthermore it has been shown repeatedly that the latter substance constitutes a reliable measure of the state of ploidy of a nucleus—a definite amount of DNA being associated with a haploid set of chromosomes.[4–7]

For the purpose of this experiment several 4-month-old Long-Evans S-1 rats were ovariectomized; 6 weeks later, some of these were injected intramuscularly with 250 I. U. of estrone in sesame oil, others with 0.5 mg. of progesterone, daily for 4 days. Vaginal smears were taken at the time of sacrifice, 6 hours after the last injection, to determine the efficacy of castration and of hormone treatment. Uterine segments were fixed in acetic alcohol. Paraffin sections of the uterus from one castrate control, one estrogen-treated and one progesterone-treated animal were mounted together on slides and treated with the Feulgen reagent for determination of DNA and the Millon reagent for determination of tyrosine and trypto-

phane. The amount of colored substance in single nuclei was determined from its specific light absorption measured with a sensitive phototube through a microscope. Techniques and methods of calculation are discussed in a previous publication.[8] The results of the DNA measurements are given in table 1.

The volume determinations incidental to these measurements confirm the observation of Salvatore, namely, that estrogen treatment caused a doubling of nuclear volumes in the uterine gland cells. Progesterone alone caused no such increase. The effect of simultaneous administration of estrogen and progesterone is still to be determined. The basic nuclear volume in the present material is slightly different from that reported by Salvatore (77.1 μ^3 instead of 88.1 μ^3). This may be due to a difference in fixation and/or to differences in the strain of rats used and is not considered to be important. The Feulgen measurements clearly indicate that the nuclei in all three animals contain essentially the same amount of DNA.

TABLE 1

AMOUNTS OF DNA (ARBITRARY UNITS) IN INDIVIDUAL NUCLEI OF UTERINE GLANDS OF THE RAT

ANIMAL	NUMBER OF NUCLEI MEASURED	NUCLEAR VOLUME IN μ^3	AMOUNT OF DNA
Castrate control	20	77.1 ± 1.0	2.41 ± 0.09
Estrogen-treated	20	157.5 ± 1.6	2.36 ± 0.08
Progesterone-treated	12	81.6 ± 1.5	2.15 ± 0.08
Tetraploid liver nuclei of a normal animal	10	[a]	5.04 ± 0.15

[a] Nuclear volume omitted, since there is often no consistent relationship among values obtained from nuclei of different tissues.[4]

Furthermore, this amount is that of a typical diploid nucleus, as is demonstrated by comparison with values obtained from tetraploid liver nuclei of a control rat measured on the same slide. Thus, injection of estrogen has not resulted in doubling of DNA content in uterine gland nuclei, and polyploidy, as would be indicated by an increased amount of DNA, evidently has not been induced by the hormone.

Doubling of nuclear volume, however, is a feature which normally accompanies polyploidy within a tissue. In mouse liver, for example, three size classes and three corresponding classes of DNA content in a ratio of 1:2:4 occur.[4, 6, 7] It has been argued[9] that the increase in nuclear volume in such a case is probably not due to the increase of DNA alone but rather to increased protein content, inasmuch as such nuclei contain anywhere from 8 to 15 times more protein than DNA.[10] This argument is supported by observations on a number of cases, where changes in nuclear volume are not accompanied by changes in DNA content but by parallel changes in

total protein content.[8, 9, 11] Table 2 indicates that this is also true in the present case.

The extinctions due to the Millon reaction are nearly the same in sections of equal thickness of small and large nuclei; this indicates that the protein concentrations in nuclei of castrate control and estrogen-treated animals are very similar. The total amounts of protein in these nuclei are accordingly correlated with the nuclear volumes, and it appears that estrogen treatment has induced an approximate doubling of the protein content in uterine gland nuclei.

This particular case of nuclear protein synthesis independent of DNA synthesis is similar to that reported by Schrader and Leuchtenberger.[9] In the testis of the hemipteran insect *Arvelius albopunctatus*, spermatocyte nuclei in different lobes of the testis stand in a volume relation of 1:2:8 but contain the same amount of DNA. The same concentration of protein was found in the medium- and large-sized nuclei. The authors then draw attention to the amazing parallelism between nuclear size and protein content and indicate their opinion that a rhythmic growth of extrachromosomal proteins is involved. The functional significance of this

TABLE 2

PROTEIN CONCENTRATIONS IN UTERINE GLAND NUCLEI AS INDICATED BY THE EXTINCTIONS (OPTICAL DENSITIES) DUE TO THE MILLON REACTION OVER 4 μ PATH LENGTH

ANIMAL	E_{TEST} (12 MEASUREMENTS)	E_{BLANK} (10 MEASUREMENTS)	$E_{CORRECTED}$ (TEST − BLANK)
Castrate control	0.157 ± 0.010	0.037 ± 0.002	0.120 ± 0.010
Estrogen-treated	0.147 ± 0.006	0.042 ± 0.003	0.105 ± 0.007

phenomenon in *Arvelius* is obscure, since both types of spermatocytes eventually give rise to apparently normal sperm cells. In the estrogen-treated rat, on the other hand, the phenomenon of rhythmic protein synthesis may be related to the function of uterine gland cells which are prepared for secretory activity by estrogen.

Protein measurements as employed herein do not permit distinction between chromosomal and extra-chromosomal proteins of the nucleus, but further attempts to specify the chemical nature of these proteins are being undertaken in order to obtain evidence bearing on the concepts of chromosome structure proposed by Mirsky and Ris.[12] Exact duplication of one or all nuclear protein components, if confirmed by more refined methods, may indicate that a multiplication of some basic units is involved. Our investigations are being extended to include the study of other instances of hormonal alteration of nuclear and cytoplasmic constituents with special reference to the anabolic activities of the nucleus.

Summary.—Our work confirms that injection of estrogen into ovariectomized rats results in a doubling of nuclear volume in uterine gland cells. Photometric analysis shows that the increase in nuclear volume is paralleled

by an increase in protein content, while the amount of desoxyribonucleic acid remains unchanged.

Acknowledgments.—This work was supported by research grants from the University of California Board of Research. We are greatly indebted to Professor A. W. Pollister, Dept. of Zoology, Columbia University, for allowing the senior author use of his laboratory facilities to conduct the measurements described herein.

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THE STRUCTURE OF PROTEINS: TWO HYDROGEN-BONDED HELICAL CONFIGURATIONS OF THE POLYPEPTIDE CHAIN

BY LINUS PAULING, ROBERT B. COREY, AND H. R. BRANSON*

GATES AND CRELLIN LABORATORIES OF CHEMISTRY,
CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA†

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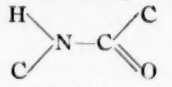
During the past fifteen years we have been attacking the problem of the structure of proteins in several ways. One of these ways is the complete and accurate determination of the crystal structure of amino acids, peptides, and other simple substances related to proteins, in order that information about interatomic distances, bond angles, and other configurational parameters might be obtained that would permit the reliable prediction of reasonable configurations for the polypeptide chain. We have now used this information to construct two reasonable hydrogen-bonded helical configurations for the polypeptide chain; we think that it is likely that these configurations constitute an important part of the structure of both fibrous and globular proteins, as well as of synthetic polypeptides. A letter announcing their discovery was published last year.[1]

The problem that we have set ourselves is that of finding all hydrogen-bonded structures for a single polypeptide chain, in which the residues are

equivalent (except for the differences in the side chain R). An amino acid residue (other than glycine) has no symmetry elements. The general operation of conversion of one residue of a single chain into a second residue equivalent to the first is accordingly a rotation about an axis accompanied by translation along the axis. Hence the only configurations for a chain compatible with our postulate of equivalence of the residues are helical configurations. For rotational angle 180° the helical configurations may degenerate to a simple chain with all of the principal atoms, C, C′ (the carbonyl carbon), N, and O, in the same plane.

We assume that, because of the resonance of the double bond between the carbon-oxygen and carbon-nitrogen positions, the configuration of each residue  is planar. This structural feature has been verified for each of the amides that we have studied. Moreover, the resonance theory is now so well grounded and its experimental substantiation so extensive that there can be no doubt whatever about its application to the amide group. The observed C—N distance, 1.32 Å, corresponds to nearly 50 per cent double-bond character, and we may conclude that rotation by as much as 10° from the planar configuration would result in instability by about 1 kcal. mole^{-1}. The interatomic distances and bond angles within the residue are assumed to have the values shown in figure 1. These values have been formulated[2] by consideration of the experimental values found in the crystal structure studies of DL-alanine,[3] L-threonine,[4] N-acetylglycine[5], and β-glycylglycine[6] that have been made in our Laboratories. It is further assumed that each nitrogen atom forms a hydrogen bond with an oxygen atom of another residue, with the nitrogen-oxygen distance equal to 2.72 Å, and that the vector from the nitrogen atom to the hydrogen-bonded oxygen atom lies not more than 30° from the N—H direction. The energy of an N—H · · · O=C hydrogen bond is of the order

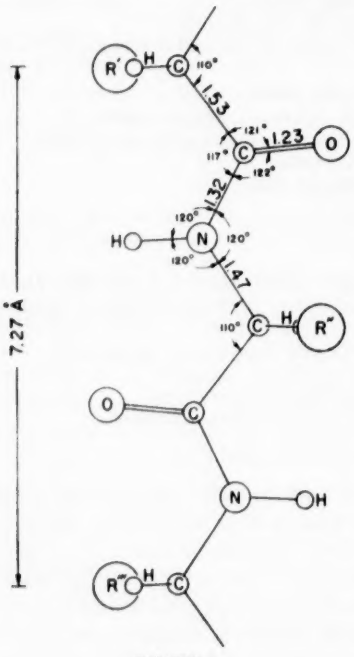


FIGURE 1

Dimensions of the polypeptide chain.

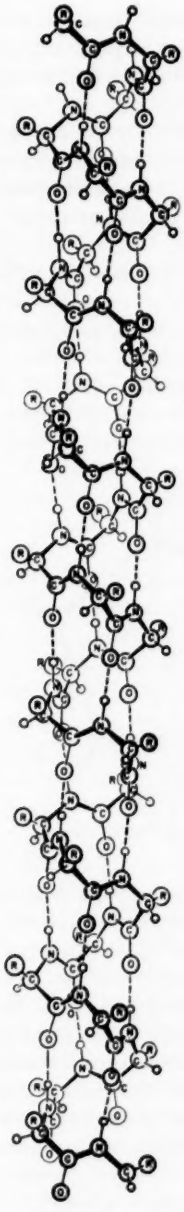


FIGURE 2
The helix with 3.7 residues per turn.

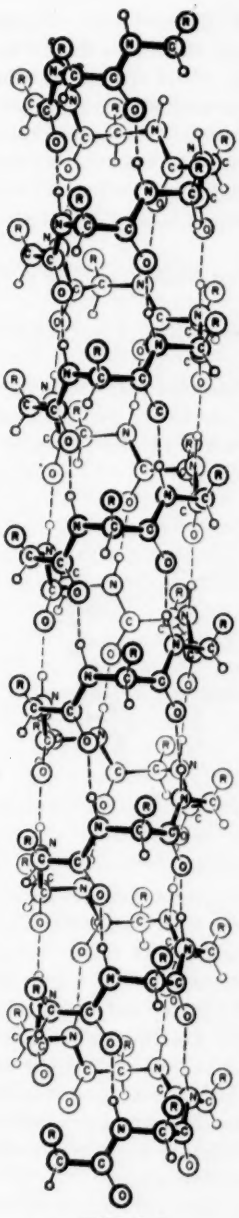


FIGURE 3
The helix with 5.1 residues per turn.

of 8 kcal. mole^{-1}, and such great instability would result from the failure to form these bonds that we may be confident of their presence. The N—H · · · O distance cannot be expected to be exactly 2.72 Å, but might deviate somewhat from this value.

Solution of this problem shows that there are five and only five configurations for the chain that satisfy the conditions other than that of direction of the hydrogen bond relative to the N—H direction. These correspond to the values 165°, 120°, 108°, 97.2° and 70.1° for the rotational angle. In the first, third, and fifth of these structures the >CO group is negatively and the >N—H group positively directed along the helical axis, taken as the direction corresponding to the sequence—CHR—CO—NH—CHR— of atoms in the peptide chain, and in the other two their directions are reversed. The first three of the structures are unsatisfactory, in that the

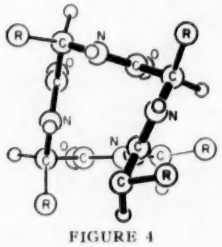


FIGURE 4

Plan of the 3.7-residue helix.

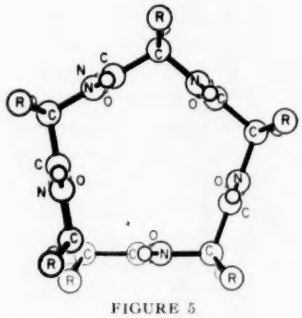


FIGURE 5

Plan of the 5.1-residue helix.

N—H group does not extend in the direction of the oxygen atom at 2.72 Å; the fourth and fifth are satisfactory, the angle between the N—H vector and N—O vector being about 10° and 25° for these two structures respectively. The fourth structure has 3.69 amino acid residues per turn in the helix, and the fifth structure has 5.13 residues per turn. In the fourth structure each amide group is hydrogen-bonded to the third amide group beyond it along the helix, and in the fifth structure each is bonded to the fifth amide group beyond it; we shall call these structures either the 3.7-residue structure and the 5.1-residue structure, respectively, or the third-amide hydrogen-bonded structure and the fifth-amide hydrogen-bonded structure.

Drawings of the two structures are shown in figures 2, 3, 4, and 5.

For glycine both the 3.7-residue helix and the 5.1-residue helix could occur with either a positive or a negative rotational translation; that is, as either a positive or a negative helix, relative to the positive direction of the helical axis given by the sequence of atoms in the peptide chain. For other amino acids with the L configuration, however, the positive helix and the negative helix would differ in the position of the side chains, and it might well be expected that in each case one sense of the helix would be more stable than the other. An arbitrary assignment of the R groups has been made in the figures.

The translation along the helical axis in the 3.7-residue helix is 1.47 Å, and that in the 5.1-residue helix is 0.99 Å. The values for one complete turn are 5.44 Å and 5.03 Å, respectively. These values are calculated for the hydrogen-bond distance 2.72 Å; they would have to be increased by a few per cent, in case that a larger hydrogen-bond distance (2.80 Å, say) were present.

The stability of our helical structures in a non-crystalline phase depends solely on interactions between adjacent residues, and does not require that the number of residues per turn be a ratio of small integers. The value 3.69 residues per turn, for the third-amide hydrogen-bonded helix, is most closely approximated by 48 residues in thirteen turns (3.693 residues per turn), and the value 5.13 for the other helix is most closely approximated by 41 residues in eight turns. It is to be expected that the number of residues per turn would be affected somewhat by change in the hydrogen-bond distance, and also that the interaction of helical molecules with neighboring similar molecules in a crystal would cause small torques in the helixes, deforming them slightly into configurations with a rational number of residues per turn. For the third-amide hydrogen-bonded helix the simplest structures of this sort that we would predict are the 11-residue, 3-turn helix (3.67 residues per turn), the 15-residue, 4-turn helix (3.75), and the 18-residue, 5-turn helix (3.60). We have found some evidence indicating that the first and third of these slight variants of this helix exist in crystalline polypeptides.

These helical structures have not previously been described. In addition to the extended polypeptide chain configuration, which for nearly thirty years has been assumed to be present in stretched hair and other proteins with the β-keratin structure, configurations for the polypeptide chain have been proposed by Astbury and Bell,[7] and especially by Huggins[8] and by Bragg, Kendrew, and Perutz.[9] Huggins discussed a number of structures involving intramolecular hydrogen bonds, and Bragg, Kendrew, and Perutz extended the discussion to include additional structures, and investigated the compatibility of the structures with x-ray diffraction data for hemoglobin and myoglobin. None of these authors proposed either our 3.7-residue helix or our 5.1-residue helix. On the other hand, we would

eliminate, by our basic postulates, all of the structures proposed by them. The reason for the difference in results obtained by other investigators and by us through essentially similar arguments is that both Bragg and his collaborators and Huggins discussed in detail only helical structures with an integral number of residues per turn, and moreover assumed only a rough approximation to the requirements about interatomic distances, bond angles, and planarity of the conjugated amide group, as given by our investigations of simpler substances. We contend that these stereochemical features must be very closely retained in stable configurations of polypeptide chains in proteins, and that there is no special stability associated with an integral number of residues per turn in the helical molecule. Bragg, Kendrew, and Perutz have described a structure topologically similar to our 3.7-residue helix as a hydrogen-bonded helix with 4 residues per turn. In their thorough comparison of their models with Patterson projections for hemoglobin and myoglobin they eliminated this structure, and drew the cautious conclusion that the evidence favors the non-helical 3-residue folded α-keratin configuration of Astbury and Bell, in which only one-third of the carbonyl and amino groups are involved in intramolecular hydrogen-bond formation.

It is our opinion that the structure of α-keratin, α-myosin, and similar fibrous proteins is closely represented by our 3.7-residue helix, and that this helix also constitutes an important structural feature in hemoglobin, myoglobin, and other globular proteins, as well as of synthetic polypeptides. We think that the 5.1-residue helix may be represented in nature by supercontracted keratin and supercontracted myosin. The evidence leading us to these conclusions will be presented in later papers.

Our work has been aided by grants from The Rockefeller Foundation, The National Foundation for Infantile Paralysis, and The U. S. Public Health Service. Many calculations were carried out by Dr. S. Weinbaum.

Summary.—Two hydrogen-bonded helical structures for a polypeptide chain have been found in which the residues are stereochemically equivalent, the interatomic distances and bond angles have values found in amino acids, peptides, and other simple substances related to proteins, and the conjugated amide system is planar. In one structure, with 3.7 residues per turn, each carbonyl and imino group is attached by a hydrogen bond to the complementary group in the third amide group removed from it in the polypeptide chain, and in the other structure, with 5.1 residues per turn, each is bonded to the fifth amide group.

* Present address, Howard University, Washington, D. C.

† Contribution No. 1538.

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CONCERNING NON-CONTINUABLE, TRANSCENDENTALLY TRANSCENDENTAL POWER SERIES

BY F. BAGEMIHL

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER

Communicated by J. L. Walsh, February 23, 1951

The main purpose of this note is to show that power series of the kind described in the title can be obtained from a given power series by simply multiplying certain of its coefficients by -1.

Consider the class $\mathfrak{K}$ of power series of the form $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ whose circle of convergence is the unit circle. There are $\mathfrak{c}$ elements in $\mathfrak{K}$ (where $\mathfrak{c}$ denotes the power of the continuum). Let $\mathfrak{C}$ be the class of those series in $\mathfrak{K}$ which can be continued beyond the unit circle, and let $\mathfrak{A}$ be the class of those series in $\mathfrak{K}$ which satisfy an algebraic differential equation. Denote by $\mathfrak{C}'$, $\mathfrak{A}'$, the respective complements of $\mathfrak{C}$, $\mathfrak{A}$, with respect to $\mathfrak{K}$.

There are the following sufficient conditions for a series in $\mathfrak{K}$ to belong to $\mathfrak{C}'$, $\mathfrak{A}'$, respectively:

(*A*)[1] *Let* $\{\lambda_\nu\}$ $(\nu = 1, 2, 3, \ldots)$ *be an increasing sequence of non-negative integers such that* $\lambda_\nu/\nu \to \infty$ *as* $\nu \to \infty$. *If* $\sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$ *belongs to* $\mathfrak{K}$, *then it also belongs to* $\mathfrak{C}'$.

(*B*)[2] *Let* $\{\lambda_\nu\}$ $(\nu = 1, 2, 3, \ldots)$ *be a sequence of non-negative integers such that* $\lambda_{\nu+1} > \nu\lambda_\nu$ *for every* ν. *If* $\sum_{\nu=1}^{\infty} a_\nu z^{\lambda_\nu}$ *belongs to* $\mathfrak{K}$, *then it also belongs to* $\mathfrak{A}'$.

The series $\sum_{\nu=0}^{\infty} z^\nu$, which represents $(1 - z)^{-1}$ for $|z| < 1$, belongs to $\mathfrak{C}\mathfrak{A}$ (i.e., to both $\mathfrak{C}$ and $\mathfrak{A}$). The series $\sum_{\nu=0}^{\infty} b_\nu z^\nu$, which represents the meromorphic function $\Gamma(z + 1)$ for $|z| < 1$, belongs to $\mathfrak{C}$ and[3] to $\mathfrak{A}'$. According to (*A*), $\sum_{\nu=0}^{\infty} z^{\nu^2}$ belongs to $\mathfrak{C}'$, and it is known[4] that this series belongs to $\mathfrak{A}$. Finally, it follows from (*A*) and (*B*) that $\sum_{\nu=0}^{\infty} z^{\nu!}$ belongs to $\mathfrak{C}'\mathfrak{A}'$. Thus,

none of the classes $\mathfrak{C}\mathfrak{A}$, $\mathfrak{C}\mathfrak{A}'$, $\mathfrak{C}'\mathfrak{A}$, $\mathfrak{C}'\mathfrak{A}'$ is empty. In fact, *each of these classes contains* $\mathfrak{c}$ *elements;* for if an arbitrary constant is added to any one of the four series just mentioned, the resulting series belongs to the same class as the original.

Let $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ belong to $\mathfrak{K}$. Call a sequence $\{\epsilon_\nu\}$ such that $\epsilon_\nu = \pm 1$ ($\nu = 0, 1, 2, \ldots$) a "sign sequence." Then[5] there is a sign sequence $\{\epsilon_\nu\}$ such that $\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$ belongs to $\mathfrak{C}'$; indeed, there are $\mathfrak{c}$ such sign sequences.[6] It is also known[7] that there are infinitely many sign sequences $\{\delta_\nu\}$ such that $\sum_{\nu=0}^{\infty} \delta_\nu a_\nu z^\nu$ belongs to $\mathfrak{A}'$.[8] We now prove the following

THEOREM. *Let* $\sum_{\nu=0}^{\infty} a_\nu z^\nu = f(z)$ *belong to* $\mathfrak{K}$. *Then there are* $\mathfrak{c}$ *sign sequences* $\{\epsilon_\nu\}$ *such that* $\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$ *belongs to* $\mathfrak{C}'\mathfrak{A}'$.

Proof:[9] Since our series, by assumption, belongs to $\mathfrak{K}$, there is an increasing sequence of natural numbers $\{\nu_\kappa\}$ ($\kappa = 1, 2, 3, \ldots$) such that, for every κ, $\nu_{\kappa+1} > \kappa\nu_\kappa$, $a_{\nu_\kappa} \neq 0$, and such that $\lim |a_{\nu_\kappa}|^{1/\nu_\kappa} = 1$ as $\kappa \to \infty$. It follows then that $\nu_\kappa/\kappa \to \infty$ as $\kappa \to \infty$. According to (A) and (B), $\sum_{\kappa=1}^{\infty} a_{\nu_\kappa} z^{\nu_\kappa} = g(z)$ belongs to $\mathfrak{C}'\mathfrak{A}'$; and the series $a \cdot g_{\mu\sigma}(z) = a \cdot \sum_{\kappa=1}^{\infty} \sigma_\kappa a_{\mu_\kappa} z^{\mu_\kappa}$, where $a \neq 0$ is an arbitrary constant, $\{\mu_\kappa\}$ is any infinite subsequence of $\{\nu_\kappa\}$, and $\{\sigma_\kappa\}$ is any sign sequence, belongs *a fortiori* to $\mathfrak{C}'\mathfrak{A}'$. Set $f(z)$-$g(z) = f_0(z)$. Divide $g(z)$ into an infinite sequence of power series $f_1(z)$, $f_2(z)$, $\ldots$, $f_\rho(z)$, $\ldots$, each of which contains infinitely many terms, such that every term of $g(z)$ belongs to precisely one $f_\rho(z)$ with $\rho \geqq 1$. Consider the set of all power series

$$F(z) = f_0(z) + \sum_{\gamma=1}^{\infty} \delta_\gamma f_\gamma(z), \qquad \delta_\gamma = \pm 1. \tag{1}$$

At most an enumerable number of these series can belong to $\mathfrak{C}$.[6] Hence, $\mathfrak{c}$ of them must belong to $\mathfrak{C}'$, and these can be divided into $\mathfrak{c}$ pairs, because $\mathfrak{c} = \mathfrak{c} + \mathfrak{c}$. If $F_1(z)$ and $F_2(z)$ are the members of any one of these pairs, then it is evident from (1) that $F_1(z) - F_2(z) = 2g_{\mu\sigma}(z)$ for suitable sequences μ and σ. As we remarked before, $2g_{\mu\sigma}(z)$ belongs to $\mathfrak{A}'$, so that[10] at least one of $F_1(z)$, $F_2(z)$ belongs to $\mathfrak{A}'$; and this completes the proof.

The theorem does not remain valid if $\mathfrak{C}'\mathfrak{A}'$ is replaced by any one of the other three classes, because, according to (A) and (B), $\sum_{\nu=0}^{\infty} \epsilon_\nu z^{\nu!}$, e.g., belongs to $\mathfrak{C}'\mathfrak{A}'$ for every sign sequence $\{\epsilon_\nu\}$, due to the presence of large gaps (i.e., consecutive terms whose coefficients are zero). *There are, moreover,* $\mathfrak{c}$ *series in* $\mathfrak{K}$, *none of which has gaps, and yet each of which belongs to* $\mathfrak{C}'\mathfrak{A}'$ *for every sign sequence.* For let $\{\epsilon_\nu\}$ be an arbitrary sign sequence. Consider

$\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$, where $a_0 = 1$, $a_\nu = \nu$ for every ν belonging to the sequence $\{\nu_\kappa\}$, where $\nu_1 = 1$, $\nu_{\kappa+1} = \nu_\kappa + \kappa$ ($\kappa = 1, 2, 3, \ldots$), whereas $a_\nu = 1/\nu^{\nu^2}$ for every other ν (this series obviously belongs to $\mathfrak{K}$). Every coefficient is different from zero, and is an algebraic number, being either an integer or the reciprocal of an integer. Furthermore, there is clearly no constant $c > 0$ such that $|\epsilon_\nu/\nu^{\nu^2}| \geqq \exp(-c\nu(\log \nu)^2)$ for every $\nu \geqq 2$. Consequently,[11] $\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$ belongs to $\mathfrak{A}'$. If ν is not a term of the sequence $\{\nu_\kappa\}$, then $|\epsilon_\nu a_\nu| \leqq 1$. The sequence $\{\epsilon_{\nu_\kappa} a_{\nu_\kappa}\}$, however, is unbounded, and $\nu_{\kappa+1} - \nu_\kappa \to \infty$ as $\kappa \to \infty$. Therefore[12] $\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$ cannot be bounded in any sector of the unit circle, which means that this series belongs to $\mathfrak{C}'$. Thus, $\sum_{\nu=0}^{\infty} \epsilon_\nu a_\nu z^\nu$ has no gaps and yet belongs to $\mathfrak{C}'\mathfrak{A}'$ for every sign sequence $\{\epsilon_\nu\}$. We can easily obtain $\mathfrak{c}$ such series, and, in fact, having rational coefficients, from $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ by replacing $a_\nu = 1/\nu^{\nu^2}$, for every $\nu \neq 0$ which does not belong to $\{\nu_\kappa\}$, by any one of the numbers $1/\tau^{\tau^2}$ ($\tau = \nu, \nu+1, \nu+2, \ldots$).

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SIMPLE WAVES IN THE COMPLETE GENERAL PROBLEM OF PLASTICITY THEORY

By Hilda Geiringer

Department of Mathematics, Wheaton College

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The Problem.—The "complete" problem of the plane perfectly plastic body is determined by five equations for five variables: the components σ_x, σ_y, τ of the symmetric plane stress tensor, Σ, and the components, v_x, v_y, of the flow velocity vector, $\bar{v}$; the system of equations is supplemented by adequate boundary conditions. One of the five equations, the "yield condition" is a finite relation between σ_x, σ_y, τ, stating that the plastic state of stress is restricted to a two-dimensional variety.

In "plane strain," $\epsilon_z = \dfrac{\partial v_z}{\partial z} = 0$, and for all other variables likewise: $\partial/\partial z = 0$. Because of the approximate incompressibility of the three-dimensional plastic medium one assumes, $\epsilon_x + \epsilon_y + \epsilon_z = 0$; hence $\epsilon_z = 0$ implies $\epsilon_x + \epsilon_y = 0$, and vice versa. With

$$\sigma_x' = -\sigma_y' = \frac{1}{2}(\sigma_x - \sigma_y), \frac{\partial v_x}{\partial x} = \epsilon_x, \frac{\partial v_y}{\partial y} = \epsilon_y, \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} = \gamma$$

the five equations for "plane strain" are[1]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \tag{1}$$

$$f(\sigma_x, \sigma_y, \tau) = F(\sigma_1, \sigma_2) = 0 \text{ "yield condition"} \tag{2}$$

$$\epsilon_x/\sigma_x' = \epsilon_y/\sigma_y' = \gamma/2\tau. \tag{3}$$

For reasons of symmetry $f(\sigma_x, \sigma_y, \tau)$ in (2) depends, in case of an isotropic medium, merely on the two principal stress components, σ_1, σ_2. From (3) we derive

$$\left(\frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial x}\right) \Big/ \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = \frac{\sigma_y - \sigma_x}{2\tau} \tag{3'}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \epsilon_x + \epsilon_y = 0 \tag{3''}$$

For "characteristic" coordinates $\alpha(x, y) = \text{const.}$, $\beta(x, y) = \text{const.}$, with v_α, v_β denoting the components of $\bar{v}$ in the α- and β-directions, (3'), (3'') may be replaced by

$$\frac{\partial v_\alpha}{\partial \alpha} - v_\beta = 0, \quad \frac{\partial v_\beta}{\partial \beta} + v_\alpha = 0 \tag{4}$$

equations which do no more contain explicitly the stress components.[2]

In the literature of plane strain a particular case is almost exclusively considered: (2) is assumed as

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 - 4K^2 \equiv (\sigma_1 - \sigma_2)^2 - 4K^2 = 0 \qquad (2')$$

where K is the yield stress in shear. We shall call the problem (1), (2'), (3) the "classical problem";[3] we shall, however, see that many results usually proved for (2') are actually valid for general plane strain.

In the typical plane problem of a thin plate under the action of forces applied to its contour parallel to the middle plane of the plate, the assumption, $\epsilon_z = 0$, *cannot* be made. If then this assumption is abandoned, (3) *must likewise be abandoned* because (3) leads to (3''). Following von Mises,[4] (3) is then replaced by

$$\epsilon_x \Big/ \frac{\partial f}{\partial \sigma_x} = \epsilon_y \Big/ \frac{\partial f}{\partial \sigma_y} = \gamma \Big/ \frac{\partial f}{\partial \tau} \qquad (5)$$

which for (2) = (2') reduces to (3). Equations (1), (2), (5) form *the general complete plane problem*. The "stress problem" (1), (2) has been considered only recently, by Sokolovsky,[5] von Mises,[6] Geiringer,[7] Sauer[8] and Hodge.[9] In the present paper we shall study the complete problem (1), (2), (5)[10] with respect to *basic equations*, *characteristics* and *initial value problems*. Finally, for "simple waves" both stresses and velocities will be explicitly determined.

General Stress Problem.—The three equations (1), (2) may be reduced in several ways to two equations. Sauer[8] uses, to advantage, the two variables σ_x, τ. Hodge[9] uses $(\sigma_1 + \sigma_2)$ and $(\sigma_1 - \sigma_2)$. Following von Mises[6] we use the variables, s and θ, where θ is the angle of the x-axis and the first principal stress direction, and s is a parameter by means of which (2) is represented as $\sigma_1 = \sigma_1(s)$, $\sigma_2 = \sigma_2(s)$. Denote by ξ, η the directions of first and second principal stress; (1), (2) are then replaced by

$$\frac{\partial s}{\partial \xi} = f(s) \frac{\partial \theta}{\partial \eta}, \quad \frac{\partial s}{\partial \eta} = g(s) \frac{\partial \theta}{\partial \xi}. \qquad (6)$$

where $\dfrac{\partial}{\partial \xi}$, $\dfrac{\partial}{\partial \eta}$ are directional derivatives and, with $\sigma_i' = \dfrac{d\sigma_i}{ds}$:

$$f(s) = \frac{1}{\sigma_1'}(\sigma_2 - \sigma_1), \quad g(s) = \frac{1}{\sigma_2'}(\sigma_2 - \sigma_1). \qquad (7)$$

The equations (6) are "reducible."[11] The plane of these variables, s, θ, is called "stress graph" by von Mises. In this plane there are *fixed characteristics*, Γ^+, Γ^-, determined by

$$\frac{d\theta}{ds} = \pm (fg)^{-1/2}, \quad \text{or} \quad \pm\theta = F(s) + \text{const.} \tag{8}$$

The characteristics, C^+ C^- in the x, y plane, the *physical plane*, depend on the solution (on the boundary conditions) of a problem. We find for them:[7]

$$\frac{d\eta}{d\xi} = \tan\varphi = \pm\sqrt{\frac{f(s)}{g(s)}}; \quad \tan^2\varphi = \frac{f}{g} = \frac{\sigma_2'}{\sigma_1'} = \frac{d\sigma_2}{d\sigma_1} \tag{9}$$

where $d\sigma_2/d\sigma_1$ follows directly from the yield condition $F(\sigma_1, \sigma_2) = 0$; also

$$m = \frac{dy}{dx} = \tan(\theta \pm \varphi). \tag{10}$$

General Velocity Problem.—From (2) with $\frac{\partial F}{\partial \sigma_i} = F_i$, $(i = 1, 2)$ we find

$$\frac{\partial F}{\partial \sigma_x} = F_1\cos^2\theta + F_2\sin^2\theta, \quad \frac{\partial F}{\partial \sigma_y} = F_1\sin^2\theta + F_2\cos^2\theta, \quad \frac{\partial F}{\partial \tau} = (F_1 - F_2)\sin 2\theta. \tag{11}$$

Because of (5), $\epsilon_x + \epsilon_y = 0$, implies $\frac{\partial f}{\partial \sigma_x} + \frac{\partial f}{\partial \sigma_y} = F_1 + F_2 = 0$, and $\tan^2\varphi = d\sigma_2/d\sigma_1 = -F_1/F_2 = 1$, $\varphi = \pi/4$, $C^+ \perp C^-$. *The characteristics, C^+, C^-, form an orthogonal net, i.e., $\varphi = \frac{\pi}{4}$, if and only if $\epsilon_x + \epsilon_y = 0$, which in turn, is a consequence of (3); the C^+, C^- then coincide with the shear lines; the general form of (2) is then $p(\sigma_1 - \sigma_2) = 0$,* or (without isotropy restriction), $\tau = q(\sigma_x - \sigma_y)$ *where p and q are arbitrary functions of one variable. If $\epsilon_x + \epsilon_y + \epsilon_z = 0$ is assumed, this is the case of "plane strain," $\epsilon_z = 0$.*

Using (5) as well as (9), (10) we obtain our *basic velocity equations*

$$\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\Big/\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = -\frac{\cos 2\varphi}{\sin 2\theta}, \quad \left(\frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial x}\right)\Big/\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) = -\cot 2\theta, \quad \frac{\partial v_x}{\partial x}\Big/\frac{\partial v_y}{\partial y} = \tan(\theta + \varphi)\tan(\theta - \varphi), \tag{12}$$

Any of the eqs. (12) follows from the two others. If the stress problem has been solved, i.e., if $\theta = \theta(x, y)$, $\varphi = \varphi(x, y)$ is known, the eqs. (12) form a *linear* system; for $\varphi = \pi/4$ they reduce to the particular eqs. (3′), (3″). It is seen that, as in the "classical" case, *the characteristics of (12) coincide with the characteristics, C^+, C^- of the stress problem (1), (2).*[10]

Let us introduce the v_x, v_y-plane, the *velocity plane* (Fig. 1). If in (12), in a domain where the respective Jacobien is not zero, we interchange de-

pendent and independent variables we find two equations whose characteristics, $\bar{C}^+$, $\bar{C}^-$ exhibit a remarkably simple relation to the C^+, C^-:

$$n = \frac{dv_y}{dv_x} = -\cot(\theta \pm \varphi) = -\frac{1}{m} \tag{13}$$

$$\bar{C}^+ \perp C^+; \bar{C}^- \perp C^- \tag{13'}$$

The relations between the C^+, C^- in the *physical plane*, Γ^+, Γ^- in the *stress plane*, $\bar{C}^+$, $\bar{C}^-$ in the *velocity plane* can be used to obtain in a very

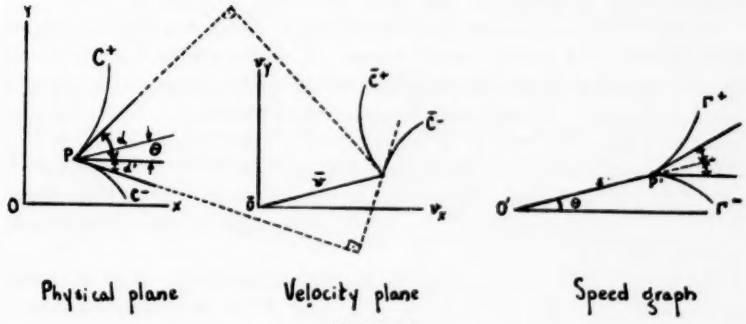

FIGURE 1

simple geometric way approximate solutions of the following classical *initial value problems:*

(1) *Along a nowhere characteristic curve, K, in the physical plane, s,* θ and v_x, v_y *are given.*

(2) v_x (or v_y) *is given along a* C^+, v_y (or v_x) *along a* C^-.

Find s, θ, v_x, v_y *in the characteristic quadrangle determined in* (1) *by K, in* (2) *by* C^+, C^-. We omit details.

The eqs. (13) have a meaning independent of "velocity plane" and "exchange of variables." They express for the eqs. (12) the relations which always exist between the derivatives of the dependent variables in a characteristic direction. Denote by d/dl^+, d/dl^- derivatives in these directions and put $2\varphi = \alpha - \alpha'$, $\alpha = \theta + \varphi$, $\alpha' = \theta - \varphi$; (13) may be written:

$$\frac{dv_y}{dl^+}\Big/\frac{dv_x}{dl^+} = -\cot\alpha, \qquad \frac{dv_y}{dl^-}\Big/\frac{dv_x}{dl^-} = -\cot\alpha' \tag{14}$$

From (14) we now derive the eqs. which generalize our basic eqs. (4). Denote by 1, 2, 3, 4 directions which make with the x-axis the angles α,

$\alpha + \frac{\pi}{2}$, α', $\alpha' + \frac{\pi}{2}$, respectively, and by v_i $(i = 1, 2, 3, 4)$ the components of v in these directions. The result is:

$$\frac{dv_1}{dl^+} = v_2 \qquad \frac{d\alpha}{dl^+}, \quad \frac{dv_3}{dl^-} = v_4 \frac{d\alpha'}{dl^-} \tag{15}$$

If $\varphi = \pi/4$ (plane strain), $3 = 2$, $1 = 4$ and (15) reduces to (4). So far (15) contains, however, *four* components of v, and *two* definite components are still to be chosen. We now leave the general problem, and apply our equations to the case of *simple waves* where we can make a natural choice of appropriate components, and solve the whole problem explicitly.

Simple Waves.—The *stress problem* of simple waves has been considered by Geiringer.[7] A simple wave[11] region, D, of the physical plane is one that is mapped onto *one* characteristic, say Γ_0^- of the stress graph; to each point, s, θ on Γ_0^- corresponds a straight characteristic, C^+, in D, carrying constant values of s and θ; at all points of D (see (8)), $\theta + F(s)$ is constant. A simple wave may be determined by giving Γ_0^- in the stress graph, and in the physical plane a set of straight lines, as the C^+. It is then simple to find to each particular C^+, or rather to its angle α, the corresponding constant solution, s, θ, of (6).

We want now to *determine the velocity distribution in D.* Since α is constant along a C^+, eq. (15[1]) with $v_1 = v_r$, and $dl^+ = dr$ gives immediately (as in "plane strain"),

$$\frac{dv_r}{dr} = 0 \tag{16}$$

Along a rectilinear characteristic C^+ the component v_r of v in direction of C^+ remains unchanged. Next, from (15[2]) with $v_2 = v_\mu$ we derive

$$\frac{dv_r}{dl^-} \cos(\alpha - \alpha') - \frac{dv_\mu}{dl^-} \sin(\alpha - \alpha') = [v_r \sin(\alpha - \alpha') + v_\mu \cos(\alpha - \alpha')] \frac{d\alpha}{dl^-}. \tag{17}$$

Since the "cross characteristics," C^-, are not orthogonal to the C^+ we introduce as coordinate system *the rectilinear C^+ and their orthogonal trajectories.* Denoting by T one particular trajectory we use as coordinates (α, r) of a point, P_1, in D, the angle, α, which the C_1^+ through P_1 makes with the x-axis, and the directed distance, $r = OP_1$ where O is the point of intersection of C_1^+ and T. Then, if ds_α denotes the line element of the orthogonal trajectory in P_1 and R the radius of curvature of T in O we find that $ds_\alpha/d\alpha = r - R$, and from (17) we obtain

$$v_r + \frac{\partial v_\mu}{\partial \alpha} = \cot 2\varphi \left[\frac{\partial v_r}{\partial \alpha} + (r - R) \frac{\partial v_\mu}{\partial r} - v_\mu \right] \tag{18}$$

which for $\varphi = \pi/4$ reduces to $v_r + \frac{\partial v_\mu}{\partial \alpha} = 0$ (see (4)). Since R, α, φ do not depend on r we find from (18)

$$\frac{\partial^2 v_\mu}{\partial r \partial \alpha} = (r - R) \frac{\partial^2 v_\mu}{\partial r^2} \cot 2\varphi \tag{19}$$

an eq. of second order for v_μ alone. It is, however, simpler to write (18) as:

$$(R - r) \frac{\partial v_\mu}{\partial r} + \tan 2\varphi \frac{\partial v_\mu}{\partial \alpha} + v_\mu = \frac{\partial v_r}{\partial \alpha} - v_r \tan 2\varphi \equiv \omega(\alpha). \tag{20}$$

Along a C^+ both v_r and $\omega(\alpha) = (R - r) \frac{\partial v_\mu}{\partial r} + \tan 2\varphi \frac{\partial v_\mu}{\partial \alpha} + v_\mu$ are constant.

It is now easy to solve the Cauchy problem, i.e., *to find the velocity distribution in a simple wave region, D, if the velocity, v, is known along a non-characteristic curve, K, across D.* Assume along K, $v_\mu = h(\alpha)$, and $v_r = k(\alpha)$ as given. From $k(\alpha)$ and from the expression of $\tan 2\varphi$ in terms of α (which is known if a particular yield condition and a particular Γ_0^- have been chosen) we find the $\omega(\alpha)$ of (20) and the problem reduces to the solution of the linear first order equation (20). Its characteristics are

$$\frac{dr}{R - r} = \frac{d\alpha}{\tan 2\varphi} = \frac{dv_\mu}{\omega(\alpha) - v_\mu} \tag{21}$$

Each of these ordinary equations contains only two variables, and is linear. The first $dr/d\alpha = (R - r) \cot 2\varphi$ is the eq. of the *cross characteristics*, C^-. If $f_1(\alpha, r) = c_1$, $f_2(\alpha, r) = c_2$ are the solutions of (21), then $G(f_1, f_2) = 0$ (where G is an arbitrary function) is the general integral of (20). To adapt the so found $v_\mu(r, \alpha)$ to the initial condition, $v_\mu = h(\alpha)$ along K, we have to express the r along K in terms of α. Details of the procedure, as well as examples must be omitted. The "characteristic initial value problem" (see p. 217, (2)) can likewise be solved explicitly.

For the general complete problem (1), (2), (5) we have thus obtained the following results: (*a*) the velocity equations (12), (14), (15); (*b*) approximate solutions of the classical initial value problems; (*c*) exact simple wave solutions. Solutions of initial value problems form the material for building up solutions of boundary value problems.

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[10] The only start in this direction known to me is by P. G. Hodge,[9] pp. 325–326.

[11] See, e.g., Courant, R., and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, Interscience Publ., New York, 1948.

CONTINUOUS GAMES

By Samuel Karlin*

California Institute of Technology and Princeton University

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In this note a new method is presented for handling continuous games. One can deduce both new theoretical results and also qualitative statements as to the nature of optimal solutions for certain classes of kernels arising in continuous games. The essential idea is to vary the spaces of strategies and study the effect this has on the game. The methods employed involve partial orderings in abstract spaces. Throughout, we use the usual terminology of the Theory of Games.

1. Let $\mathcal{E}$ and γ denote Banach spaces consisting of measures and let $\mathfrak{R}$ and $\mathcal{S}$ denote generally spaces of functions such that $\mathfrak{R}^*$ and $\mathcal{S}^*$ (conjugate Banach space) consist again of measures. Let K denote a closed cone of elements in $\mathfrak{R}$, i.e., if $x, y \in K$, then $\lambda x + \mu y \in K$ for $\lambda, \mu \geq 0$, and if $x \in K$ and $-x$ is in K, then $x = 0$. Similarly, P shall denote a closed cone in $\mathcal{E}$. The dual cone K^* in $\mathfrak{R}^*$ shall be defined as the set of all f in $\mathfrak{R}^*$ with the property that $(x, f) \geq 0$ for every x in K. [(x, f) is the value of the functional f acting on the element x. Throughout this paper the inner product notation shall be used.] With respect to any cone one can define a partial ordering, e.g., $x \geq 0$ if and only if $x \in K$. Similarly $y \geq x$ shall mean $y - x \geq 0$. Also, let Q be a closed cone contained in K^*. We assume that the cone K possesses an interior point u. A and B shall denote bounded linear operators mapping $\mathcal{E}$ into $\mathfrak{R}$, and the conjugate mappings shall take $\mathfrak{R}^* = \gamma$ into $\mathcal{S}$. K_u^* shall denote the section of the cone K^*, where $(u, f) = 1$. Q_u is defined analogously with the section taken relative to Q. Finally, it

is assumed that in S a cone L is given and that P is contained in L^*. Also, L has an interior point v to which is associated P_v.

Definition 1: Let $\Omega_s(A, B)$ be the set of real numbers λ for which there exists a f in P_v such that

$$Af \geq \lambda Bf \qquad (Af - \lambda Bf \text{ is in } K).$$

Put $\bar{\lambda}_s = \bar{\lambda}_s(A, B) = \sup \lambda$ for λ in Ω_s.

Definition 2: $\underline{\lambda}_s$ and $\underline{\Omega}_s$ are defined in a similar manner with the above inequality reversed and sup replaced by inf.

Definition 3: Let $\Omega_\omega(A, B) = [\lambda \mid$ for any $g_1, \ldots, g_n \in Q_u$ there exists a f in P_v with $(g_i, Af) \geq \lambda(g_i, Bf) i = 1, \ldots, n]$. Clearly, f may depend upon g_i. Put $\bar{\lambda}_\omega = \sup \lambda$ for λ in Ω_ω. Similarly, $\underline{\Omega}_\omega$ and $\underline{\lambda}_\omega$ are defined.

Definition 4: It is furthermore assumed that P and Q are chosen large enough. More precisely, if $(g, x) \geq 0$ for every g in Q_u, then x is in K. The set Q_u is said to be sufficient if whenever x is of the form $x = Af - \lambda Bf$ for f in P_v and $(g, x) \geq 0$ for every g in Q_u, then x is in K. An analogous statement applies to P.

Definition 5: $\Omega_s(A^*, B^*)$, $\underline{\Omega}_s(A^*, B^*) \ldots$ and $\underline{\lambda}^*_s = \underline{\lambda}_s(A^*, B^*) \ldots$ are defined analogously in terms of the adjoint mappings and Q_u.

The meaning of these quantities is explained in the following result: If B is assumed to be strictly positive in the sense that for any $g \in Q_u$ and $f \in P_v$ it implies $(g, Bf) > 0$, then one gets with sufficiency that

$$\sup_{f \in P_v} \inf_{g \in Q_u} \frac{(g, Af)}{(g, Bf)} = \bar{\lambda}_s$$

$$\inf_{g \in Q_u} \sup_{f \in P_v} \frac{(g, Af)}{(g, Bf)} = \underline{\lambda}^*_s.$$

The assumption on B is not really essential but is made here to avoid unnecessary writing. These formulae indicate the connection with the fundamental quantities occurring in the Theory of Games. The general relationship between these quantities is given in the following Theorem. We assume the requirement of sufficiency is fulfilled.

THEOREM 1. $\bar{\lambda}_\omega \geq \underline{\lambda}^*_s \geq \bar{\lambda}_s \geq \underline{\lambda}^*_\omega$.

In the sequel, the statement that a value exists shall be synonymous with the equality of $\underline{\lambda}^*_s = \bar{\lambda}_s$. It is to be understood that the value exists means here relative to the cones P and Q.

Any one of the following conditions will insure $\underline{\lambda}^*_s = \bar{\lambda}_s$.

THEOREM 2. *(a) If the image of $Af - \lambda Bf$ for f in P_v and any fixed λ constitute a weakly compact set relative to Q_u, then this implies that $\bar{\lambda}_\omega = \bar{\lambda}_s$.*

(b) If the image of $A^*g - \lambda B^*g$ with g in Q_u is weakly compact relative to P_v, then $\underline{\lambda}^*_s = \underline{\lambda}^*_\omega$.

Remark: A set H of elements is said to be weakly compact relative to a

set G of functionals if under the weak topology on H induced by G, then H constitutes a bicompact set.

Many other slight variations of the conditions of Theorem 2 can be given which will imply the existence of a value. The advantage of this general formulation is the great flexibility in choosing the cones. It is clear that if $P_v^{(1)} \subset P_v^{(2)}$ and $Q_u^{(1)} \subset Q_u^{(2)}$ imply $\bar{\lambda}_s^{(1)} \leq \bar{\lambda}_s^{(2)} \leq \lambda_s^{*(2)} \leq \lambda_s^{*(1)}$. Thus to secure a value one wants to increase the cones P and Q sufficiently until one obtains $\bar{\lambda}_s = \underline{\lambda}_s^*$. On the other hand, if one has a value, then to obtain more refined qualitative results it is desirable to reduce the size of the cone to a minimum cone which still has a value. Furthermore, keeping Q_u fixed and decreasing P_v still maintains a value as long as P is a cone sufficient in the sense of Definition 4. Some applications of these ideas will be presented in Section 2. Also an equivalence theory for solution can be developed based on this principle. Two cones $P^{(1)}$ and $P^{(2)}$ are said to be equivalent relative to the operators A and B if $\bar{\lambda}_s(P^{(1)}) = \bar{\lambda}_s(P^{(2)})$.

Finally, in this section some results on perturbation of games and its applications are stated.

THEOREM 3. *(a) If $A_n \rightrightarrows A$ (uniformly) and $B_m \rightrightarrows B$, then $\bar{\lambda}_s^n \to \bar{\lambda}_s$ and $\underline{\lambda}_s^{*m} \to \underline{\lambda}_s^*$. The cones P and Q are held fixed.*

(b) If $A_1 \geq A_2 \geq \ldots \to A$ (strongly) and $A_n^* \to A^*$ strongly with $B_n = B$, then if A_n maps $\mathcal{E}$ into $\mathcal{R}$ and $\mathcal{E}$ is the conjugate space to the space containing the image of A_n^*, then A has a value. The cones P and Q are held fixed.

(c) If $P^1 \subset P^2 \subset P^3 \ldots \to P$ (strongly) while Q^k also increases to Q and if with respect to each P_v^n, Q_u^m a value exists, then a value also exists relative to the cones P and Q.

A few examples will illustrate this theory.

(1) Let $Af = \int M(x, y)f(x)dy$, where $M(x, y)$ is a measurable function of both variables and bounded on the unit square. Let $B = (v, f)u$, where $u = v$ is the function identically 1. Finally, let P represent the cone consisting of all absolutely continuous distributions over the interval $(0, 1)$. With the aid of Theorem 3 it can be shown if $Q = P$, then a value exists. In this case no optimal solutions in general exist, however, the value may be approached within ϵ by suitable choice of strategy.

(2) If $M(x, y)$ is a lower semicontinuous function, then a value exists relative to all completely additive distributions on the interval $(0, 1)$. The definitions of A and B are taken to be the same operators as in the previous example. This example was first achieved by I. Glicksberg.

2. In this section we indicate how qualitative results may be obtained by selecting the cones appropriately. The definitions of A and B are given as in example (1) above. The kernel $M(x, y)$ is in addition assumed to be measurable in each variable for fixed choice of the other variable. We discuss several examples.

(1) Let $M_{\nu\mu}(x, y) \geq 0$ with $M_{\nu\mu}$ being continuous in both variables.

In other words, $M(x, y)$ for each x is convex in y. We choose for P the closure of the set of all distributions which have only a finite number of points of increase. The space $\mathfrak{R}$ is taken to be the set of all functions possessing continuous second derivatives. The cone K is chosen to be all functions for which $\omega''(y) \geq 0$, $\omega'(0) \geq 0$, $\omega(0) \geq 0$. The condition $Af \geq \lambda Bf$ is equivalent with

$$\int_0^1 M_{yy}(x, y)df(x) \geq 0$$
$$\int_0^1 M_y(x, 0)df(x) \geq 0$$
$$\int_0^1 M(x, 0)df(x) \geq \lambda.$$

The first inequality imposes no restriction on any distribution. This follows from the hypothesis put upon $M(x, y)$. Thus it may be discarded and it can be concluded from the last two inequalities that an optimal solution distribution for player 1 exists using at most two jumps.

(2) If $M_{y \ldots y}(x, y) \geq 0$ with the derivative taken n times, then an optimal solution distribution for player 1 can be found with at most n points of increase.

(3) Let $M(x, y) = \begin{cases} N(x, y) & x \leq y \\ 0 & x > y \end{cases}$ and $N_y(x, y) \geq 0$.

Again, let K be chosen as the set of all continuously differentiable functions with $\omega'(y) \geq 0$ and $\omega(0) \geq 0$. Let Q be chosen to be all possible absolutely continuous distributions spread over the set Γ of y where $N(y, y) \leq 0$ and a possible jump at the origin. The condition $Af - \lambda Bf \geq 0$ becomes

$$N_y(0, y) + \int_0^y N_y(x, y)f(x)dx + N(y, y)f(y) \geq 0$$
$$\alpha N(0, y) \geq \lambda.$$

The first inequality is automatically satisfied over the set of y where $N(y, y) \geq 0$. Thus the set Q designated above becomes sufficient for this problem. Therefore, it can be shown that within ϵ an optimal solution for player 2 can be found in the cone Q.

Other types of kernels of this kind with a discontinuity along the diagonal can be handled in such a manner.

(4) If $M(x, y)$ is analytic for each x in y, then a suitable ordering for K is to require that $\omega^{(n)}(0) \geq 0$ for every n. Special results can be obtained in this circumstance.

These examples serve to illustrate the method. In essence, the idea is to utilize the properties of the kernel $M(x, y)$ and to accordingly select a cone within which a solution can be found.

3. The entire above procedure and method can be applied to the theory of linear programming. There the concept of product operators and product spaces are used. Moreover, many applications of the above theory can be found in the theory of Decision Functions and the problems of testing hypothesis. Details of the above result and additional applications will be published later.

* O. N. R. Fellow.

THEORY OF TURNS AND SLIDES UPON A SURFACE

BY EDWARD KASNER AND JOHN DE CICCO

DEPARTMENTS OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, AND DE PAUL UNIVERSITY, CHICAGO

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1. Consider a surface Σ for which the linear element is

$$ds^2 = E(x, y)dx^2 + 2F(x, y)dx\,dy + G(x, y)dy^2 \qquad (1)$$

such that $E > 0$, $G > 0$, $H > 0$, F, $H^2 = EG - F^2 > 0$ are real analytic functions over a certain region of the parametric (x, y)-plane. Thus Σ is represented in a point-to-point fashion upon this region of the plane.

A *dilatation* D_s on Σ is a lineal element transformation whereby any lineal element (x, y, p) corresponds to a lineal element (X, Y, P), such that both are orthogonal to the same geodesic through the points of the elements, and the geodesic distance between the given points is the constant number s.

The set of dilatations D_s on Σ form a one-parameter group. The infinitesimal lineal element transformation, which generates this one parameter group, is the infinitesimal dilatation $D_{\delta s}$.

The infinitesimal dilatation $D_{\delta s}$ is given by the equations

$$\begin{aligned}
H(E + 2Fp + Gp^2)^{1/2}\ \delta x &= (F + Gp)\delta s,\\
H(E + 2Fp + Gp^2)^{1/2}\ \delta y &= -(E + Fp)\delta s,\\
2H^2(E + 2Fp + Gp^2)^{1/2}\ \delta p &= \left[-H\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right)(E + 2Fp + Gp^2) + 2(E + 2Fp + Gp^2)\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial y}\right)H\right]\delta s. \qquad (2)
\end{aligned}$$

An infinitesimal dilatation upon Σ is an infinitesimal contact transformation with the characteristic function

$$W = \frac{1}{H}(E + 2Fp + Gp^2)^{1/2}. \qquad (3)$$

The dilatations upon Σ form a one-parameter group of contact transformations whose path curves are the geodesics of Σ.

Two infinitesimal contact transformations, neither of which are extended point transformations, such that they possess the same transversality law and their commutator is an extended point transformation, which is not the identity, are two infinitesimal dilatations of two conformally equivalent surfaces.

Two infinitesimal contact transformations possess the same linear involutorial law of transversals if and only if they are two infinitesimal dilatations of two conformally equivalent surfaces.[1]

2. A *turn* T_α upon a surface Σ is a lineal element transformation whereby

each lineal element is rotated about its point through a constant angle α.

The turns T_α form a one-parameter group generated by the infinitesimal turn $T_{\delta\alpha}$, namely

$$\delta x = 0,\ \delta y = 0,\ H\delta p = (E + 2Fp + Gp^2)\delta\alpha. \qquad (4)$$

The path series are the ∞^2 stars, each consisting of the ∞^1 lineal elements through a given point of Σ.

A *slide* S_k upon a surface Σ is a lineal element transformation whereby, to each lineal element (x, y, p) of Σ, there corresponds a lineal element (X, Y, P) of Σ, such that it is tangent to the geodesic of Σ determined by (x, y, p), and the geodesic distance between the points of these given lineal elements is the constant k.

The slides S_k form a one-parameter group generated by the infinitesimal slide $S_{\delta k}$, namely

$$\begin{aligned}
&(E + 2Fp + Gp^2)^{1/2}\delta x = \delta k,\\
&(E + 2Fp + Gp^2)^{1/2}\delta y = p\delta k,\\
&2H^2(E + 2Fp + Gp^2)^{1/2}\delta p = [(FE_x - 2EF_x + EE_y) +\\
&\quad (GE_x - 2EG_x - 2FF_x + 3FE_y)p + (-3FG_x + 2GE_y - EG_y +\\
&\qquad 2FF_y)p^2 + (-GG_x + 2GF_y - FG_y)p^3]\delta k. \qquad (5)
\end{aligned}$$

The path series are the ∞^2 geodesics.

3. *If U_1F, U_2F U_3F, denote the symbols of the infinitesimal turn, slide, dilatation, then their commutators are related as follows:*

$$(U_1, U_2)F = U_3F,\ (U_2, U_3)F = KU_1F,\ (U_3, U_1)F = U_2F, \qquad (6)$$

where K denotes the Gaussian curvature.

By means of these formulae, the following result is obtained.

The turns and slides upon a surface Σ form a three-parameter group, called the whirl group, if and only if Σ is of constant Gaussian curvature.

If a surface Σ is of variable Gaussian curvature, then the turns and slides upon Σ do not generate a group of any finite number of parameters.[2]

[1] Kasner, *Differential Geometric Aspects of Dynamics*, Princeton Colloquium, *T. Am. Math. Soc.*, 1913, 1949; Lie, "Die infinitesimale Berührungstransformationen der Mechanik," *Leipziger Berichte*, 145–153 (1889); Kasner, "The Infinitesimal Contact Transformations of Mechanics," *Bull. Am. Math. Soc.*, **16,** 408–412 (1910); De Cicco, "New Proofs of the Theorems of Kasner Concerning the Infinitesimal Contact Transformations of Mechanics," *J. Math. Phys.*, **26,** 104–109 (1947).

[2] The geometry of turns and slides was initially studied by Kasner in 1911, followed by a series of papers by Kasner and De Cicco. Recently, contributions to this subject have been made by A. Narasinga Rao, K. Strubecker, and J. M. Feld. See Kasner, "The Group of Turns and Slides and the Geometry of Turbines, *Am. J. Math.*, **33,** 193–202 (1911); De Cicco, "The Geometry of Fields of Lineal Elements," *Trans. Am. Math. Soc.*, **47,** 207–229 (1940). This article contains a bibliography of the subject. Feld, "A Kinematic Characterization of Series of Lineal Elements in the Plane and of Their Differential Invariants Under the Group of Whirl-Similitudes and Some of Its Subgroups," *Am. J. Math.*, **70,** 129–138 (1948).

PROBABILISTIC GEOMETRY

By Karl Menger

Illinois Institute of Technology

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We generalize the concept of a metric space by associating a cumulative distribution function, Δ_{ab}, with every ordered pair (a, b) of elements of a set S. The value $\Delta_{ab}(x)$ may be interpreted as the probability that the distance from a to b be $< x$. But S need not be a metric space in the ordinary sense. The distribution functions and the association of these functions with the pairs of elements of S, are all that is assumed. For instance, a probabilistic metric consisting of five elements is a 5-by-5 matrix of distribution functions—as an ordinary metric space consisting of five points is a 5-by-5 matrix of numbers.

In analogy to the assumptions $1'.$, $2'.$, $3'.$ about a metric space,* the non-decreasing functions Δ_{ab}, which are continuous to the right, are assumed to satisfy the following postulates. (Probabilistic interpretations are mentioned parenthetically.)

$1_a''$. $\Delta_{aa}(x) = 1$ for every point a and every $x > 0$. (The distance from a point to itself is certainly ≤ 0.)

$1_b''$. $\Delta_{ab}(x) = 0$ for every two points and every $x \leq 0$. (Negative distances are impossible.)

$2''$. $\Delta_{ab}(x) = \Delta_{ba}(x)$ for every two points and every x. (The distance from a to b is as likely to be $< x$ as is the distance from b to a.)

The triangle inequality included in our original set of postulates[1] will be replaced by the following more stringent version due to Wald.[2]

$3''$. $[\Delta_{ab} * \Delta_{bc}](x) \leq \Delta_{ac}(x)$ for every three points and every x. Here, $\Delta' * \Delta''$ denotes the convolution of Δ' and Δ'', that is, the distribution of the sum of the random variables with the distributions Δ' and Δ''. [For every x, it is less likely (or, at any rate, not likelier) that the sum of the distances from a to b and from b to c be $< x$, than that the distance from a to c be $< x$.]

An ordinary metric space with the distance function $d(a, b)$ is a special case of a probabilistic metric, namely, the case that

$$\Delta_{ab}(x) = \begin{cases} 0 \text{ if } x \leq d(a, b) \\ 1 \text{ if } x > d(a, b). \end{cases}$$

Normal distributions are ruled out by Postulate $1_b''$.

We call a and a' *certainly-indistinguishable* if $\Delta_{aa'}(x) = 1$ for each $x > 0$. Uniting all elements which are certainly indistinguishable from each other into identity sets we decompose the space into disjoint sets $A, B, \ldots$. We may define $\Delta_{AB}(x) = \Delta_{ab}(x)$ for any a belonging to A, and any b be-

longing to B. (The number is independent of the choice of a and b.) The identity sets form a perfect analog of an ordinary metric space since they satisfy the condition

$1_c''$. If $A \neq B$, then there exists a positive x such that $\Delta_{AB}(x) > 1$.

Replacing, if necessary, points by identity sets we shall assume that no two points are certainly-indistinguishable unless they are identical.

Let $\delta(a, b)$ denote the greatest lower bound of all numbers x such that $\Delta_{ab}(x) = 0$. If this certainly-lower-distance is positive, then a and b will be said to be *certainly-distinguishable*. If $\delta(a, b) = 0$, we distinguish two cases: we call a and b *barely-distinguishable* if $\Delta_{ab}(+0) = 0$; and *perhaps-indistinguishable* if $\Delta_{ab}(+0) > 0$. Here $\Delta_{ab}(+0)$ denotes the right-side limit of Δ_{ab} at 0.

The relation of being perhaps-indistinguishable is reflexive, symmetrical and (by virtue of $3''$.) transitive. Again the space can be decomposed into disjoint lumps consisting of perhaps-indistinguishable elements. But if a and a' belong to the same lump, and b is any element, the functions Δ_{ab} and $\Delta_{a'b}$ need not be equal. Even if the distances between lumps are defined by distribution functions rather than numbers, the definition of these functions must be based on averaging processes.

Now let S be an ordinary metric space with the distance function $d(a, b)$. We call a probabilistic metric *monotonic* on S if $d(a, b) \leq d(a, c)$ implies $\Delta_{ac}(x) \leq \Delta_{ab}(x)$ for every a, b, c, x. Clearly, in a monotonic metric, $d(a, b) = d(a, c)$ implies $\Delta_{ab} = \Delta_{ac}$.

If the metric space is complete, convex and externally convex[3] (so that every two points are on at least one straight line), then it can be shown that either each lump contains only one point or that there is only one lump, namely, the entire space. More precisely, we obtain the following theorem. *One of three cases is present: either every two points of S are perhaps-indistinguishable; or every two points are barely-distinguishable; or every two points are certainly-distinguishable.*

Each of these three cases can occur on the straight line, even in conjunction with the assumption of

Homogeneity. $\Delta_{bc}(x) = \Delta_{0,c-b}(x)$ for every b, c, x.

In a homogeneous metric of the line it is sufficient to define $\Delta_{0a}(x)$ for $a > 0$ and $x > 0$.

With the following quasi-exponential distributions, every two points are perhaps-indistinguishable.

$$\Delta_{0a}(x) = 1 - \Phi(a)e^{-x} \quad (a > 0, x > 0).$$

Here, Φ must be a nowhere decreasing function for which

$$\Phi(2a) \leq 2\Phi(a) - \Phi^2(a).$$

A solution satisfying the postulates is $\Phi(a) = 1 - e^{-a}$.

Exponential distributions, $\Delta_{ab}(x) = 1 - e^{-\varphi(a, b)x}$, are impossible on the

straight line (and in much more general spaces) even if the assumption of homogeneity is dropped.

With the following rectangular distributions, every two points are barely-distinguishable.

$$\Delta_{0a}(x) = x/\varphi(a) \text{ for } 0 < x \leq \varphi(a),$$

where $\varphi(a) + \varphi(b) - \varphi(a + b) \geq \varphi(a)\varphi(b)/2\varphi(a + b)$. A solution satisfying the postulates is $\varphi(a) = a^k$ if

$$1 < 2^k \leq 1 + 1/\sqrt{2} \quad \text{or} \quad 0 < k \leq 0.7714 \ldots$$

A particularly simple example is $\varphi(a) = \sqrt{a}$.

With the following rectangular distributions, every two points are certainly-distinguishable.

$$\Delta_{0a}(x) = \frac{x - \psi(a)}{\varphi(a) - \psi(a)} \text{ for } \psi(a) \leq x \leq \varphi(a),$$

where

$$\varphi(a) + \varphi(b) - \varphi(a + b) \geq \frac{[\varphi(a) - \psi(a)]\,[\varphi(b) - \psi(b)]}{2[\varphi(a + b) - \psi(a + b)]}.$$

The postulates are satisfied if we set $\varphi(a) = a^k$ $(0 < k \leq 0.7714 \ldots)$ as in the preceding example, and $\psi(a) = c\varphi(a)$ for some constant $c < 1$.

The metrics of the last two examples contract the scale in the sense that the distance from 0 to a is certainly $\leq \varphi(a)$, for instance, $\leq \sqrt{a}$. We are reminded of the psychophysical contractions of the scale, especially of Cramer's suggestion[4] that the subjective value of an amount of money, a, does not exceed $\sqrt{a}$. The assumption that the distance from 0 to a be certainly $\leq \log a$ (that is, the analog of the well-known Bernoulli-Fechner contraction) contradicts the triangle inequality for $a < 2^{\sqrt{2}}$.

An inhomogeneous metric, which does not contract the scale is defined by the following rectangular distributions

$$\Delta_{bc}(x) = \Delta_{cb}(x) = \Delta_{0,\,(c-b)c/(c-b/2)}(x) \text{ for } b \leq c \text{ and every } x,$$
$$\Delta_{0a}(x) = x/a \text{ for } 0 < x \leq a.$$

In this metric, every two points are barely-distinguishable.

For some purposes, none of the three main cases is desirable. For instance, none of them is compatible with the existence of a point a, which lies in a bounded threshold consisting of points that are perhaps-indistinguishable from a, while the points outside are certainly (or at least barely) distinguishable from a. From our theorem it follows that, in order to cover this and similar situations, we have to weaken, if not to give up, the triangle inequality.

It appears that *in the transition from the rigid geometry in the large to a*

probabilistic geometry in the small, the geometric laws to which we are accustomed, not only cease to be certain but, in some cases, become unlikely and even impossible.

There seem to exist two or three main types of applications of a probabilistic geometry.

1. To the *psychophysical continuum.* The observer starts with an objective metric space (or what, in a first approximation, he considers as a metric space): the skin of the blindfolded observed person, the sets of weights read on a scale, temperatures, and the like. Upon these objective rigid metrics, the observed person superimposes his subjective probabilistic metric.

2. To the *physical macrocosm.* Our theory provides us with a scheme for dealing with bundles of observations of spatial quantities, weights, temperatures, etc. Such a scheme should be useful where the enormously simplifying method of replacing these bundles by single numbers would prove to be an oversimplification. Underlying the (more or less arbitrary) averaging and selection procedures of the simplifying method is the idea that various phases of the macrocosm have the structure of a numerical continuum (for the elements of which we have to discover the "real" values and distances). But the foundation of this idea seems to lie only in scientific usage.

3. Perhaps to the *physical microcosm.* In this application, the fundamental difficulty seems to lie in the lack of an objective metric comparable to that of the observer of another person's psychophysical continuum. The postulate that each world point can be characterized by four numerical coordinates is not more plausible than the assumption that the distance between two points be a number. Consequently, the theory would have to develop criteria for identifying the elements of a set, along with the distance distributions between the elements.

* See my note "Probabilistic Theories of Relations" in the preceding issue of these PROCEEDINGS, **37,** 178, 1951, where the assumptions 1.′, 2.′, 3.′ are quoted on p. 179.

[1] PROC. NATL. ACAD. SCI., **28,** 535 (1942).

[2] *Ibid.*, **29,** 196 (1943).

[3] "Untersuchungen ueber allgemeine Metrik," *Math. Ann.*, **100,** 75–163 (1928).

[4] Cf. the author's paper in *Z. Nationaloekonomie*, **5,** 465 (1934).

NUMERICAL RESULTS FOR PRESSURE-VELOCITY CORRELATIONS IN HOMOGENEOUS ISOTROPIC TURBULENCE

By D. Nelson Limber

Yerkes Observatory, Williams Bay, Wisconsin, University of Chicago

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Introduction.—In a recent paper Batchelor[1] has considered fluctuations in pressure in homogeneous isotropic turbulence by studying the correlation, $\overline{pp'}$, of the pressures p and p' at two points separated by a distance r. He has shown that this correlation, which in homogeneous isotropic turbulence is a function only of r, can be expressed in terms of the scalar, Q, defining the fundamental correlation term $\overline{u_i u_j'}$ if the assumption is made that the fourth order correlation $\overline{u_i u_j u_l' u_m'}$ is related to the second order correlation $\overline{u_i u_j'}$ as if u_i, u_j and u_l', u_m' were jointly normally distributed. Batchelor's final result is that

$$\frac{1}{\rho^2}\overline{pp'} = 8\int_r^\infty (Q')^2\left(y - \frac{r^2}{y}\right)dy.$$

It is the object of this note to show that with the same assumptions as Batchelor has made, the correlation $\overline{Pu_l'u_j'}$ can also be expressed in terms of Q.

Theoretical Development.—Multiplying the Stokes-Navier equation

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i \tag{1}$$

by $u_l'u_m'$ and averaging we obtain

$$\overline{u_l'u_m'\frac{\partial u_i}{\partial t}} + \nu\nabla^2(\overline{u_i u_m u_l'}) = \frac{\partial}{\partial \xi_j}\overline{(u_i u_j u_l' u_m')} + \frac{\partial P_{lm}}{\partial \xi_i}, \tag{2}$$

where $\xi_i = x_i' - x_i$ and where P_{lm} denotes the correlation tensor $\overline{Pu_l'u_m'}$. Under conditions of isotropy P_{lm} must be of the form

$$P_{lm} = P_1\xi_i\xi_m + P_2\delta_{lm}, \tag{3}$$

where P_1 and P_2 are two even functions of the distance r separating the points considered.

On the assumption that the fourth order correlations are related to the second order correlations in the manner of a joint normal distribution,

$$\overline{(u_i u_j u_l' u_m')} = Q_{il}Q_{jm} + Q_{im}Q_{jl} + Q_{ij}(0)Q_{lm}(0) \tag{4}$$

where it is known that[2]

$$Q_{ij} = \overline{u_i u_j'} = \frac{Q}{r}\,'\xi_i\xi_j - (rQ' + 2Q)\delta_{ij}. \tag{5}$$

Since the quantity on the left-hand side of equation (2) is solenoidal in i and symmetrical in l and m, the same must be true of the terms on the right. The term

$$X_{lmi} = \frac{\partial}{\partial\xi_l}(Q_{il}Q_{jm} + Q_{im}Q_{jl}) + \frac{\partial P_{im}}{\partial\xi_l} \tag{6}$$

must therefore be solenoidal in i and symmetrical in l and m, in other words it must be derivable in terms of a single scalar X in the manner[3]

$$X_{lmi} = \text{curl } X(\xi_l\,\epsilon_{mik}\,\xi_k + \xi_m\,\epsilon_{lik}\xi_k) = \frac{2}{r}X'\xi_i\xi_m\xi_l -$$

$$(rX' + 3X)\,(\xi_l\delta_{mi} + \xi_m\delta_{il}) + 2X\delta_{lm}\xi_i. \tag{7}$$

The condition that equation (6) be of the form of equation (7) suffices to determine P_1 and P_2 in terms of Q. We find

$$r^2P_1 + 3P_2 = -4\int_r^\infty (Q')^2 y\,dy \tag{8}$$

$$\frac{P_2'}{r} = P_1 + 2(Q')^2 - 8\int_r^\infty (Q')^2\frac{1}{y}\,dy \tag{9}$$

$$P_1 = \frac{24}{5}\int_r^\infty (Q')^2\frac{1}{y}\,dy + \frac{14}{5r^5}\int^r (Q')^2 y^4\,dy. \tag{10}$$

The derivation of these results follows the development contained in a recent paper by Chandrasekhar on turbulence in magneto hydronamics.[3] Equation (8) is of special interest since by equation (3)

$$P_{ii}(r) = \frac{1}{\rho}\overline{P(u')^2} = r^2P_1 + 3P_2. \tag{11}$$

Thus

$$P_{ii}(0) - P_{ii}(r) = -4\int_0^r (Q')^2 y\,dy. \tag{12}$$

Results.—It is of special interest to see what form equation (12) takes when the Kolmogorof Region is considered. We then have[4]

$$2\overline{u^2}\left(1 - 2\frac{Q}{\overline{u^2}}\right) = c(\nu\epsilon)^{1/3}\left(\frac{r}{\eta}\right)^{2/3}, \tag{13}$$

and

$$Q' = \frac{1}{6}\frac{c(\nu\epsilon)^{1/3}}{\eta^{1/3}}\,r^{-1/3}. \tag{14}$$

Substituting the latter into equation (12) we obtain

$$\frac{1}{\rho}\left(\overline{p(u)^2} - \overline{p(u')^2}\right) = \frac{1}{12} c^2 \nu\epsilon \left(\frac{r}{\eta}\right)^{4/3}, \tag{15}$$

or by equation (13)

$$\frac{1}{\rho}\left(\overline{p(u)^2} - \overline{p(u')^2}\right) = 12\overline{(u_\parallel - u_\parallel')^2}^2. \tag{16}$$

For the correlation $\overline{pp'}$ Batchelor[1] has obtained the following relation under the same conditions:

$$\frac{1}{\rho^2}\left(\overline{p^2} - \overline{pp'}\right) = \frac{1}{2} c^2 \nu\epsilon \left(\frac{r}{\eta}\right)^{4/3}. \tag{17}$$

Thus we have the simple relation

$$\frac{1}{\rho}\left(\overline{p(u)^2} - \overline{p(u')^2}\right) = \frac{1}{6p^2}\left(\overline{p^2} - \overline{pp'}\right). \tag{18}$$

The longitudinal and transverse correlations of P_{lm} are, respectively,

$$\overline{p(0, 0)u_x(x, 0)u_x(x, 0)} = P_1(r)r^2 + P_2(r). \tag{19}$$

and

$$\overline{p(0, 0)u_y(x, 0)u_y(x, 0)} = P_2(r). \tag{20}$$

Now Proudman[5] has tabulated Q and Q' for Reynolds numbers 12.5 and ∞. His tabulations are in terms of a dimensionless variable, y, given by

$$r = \sqrt{\alpha\nu t}\, y \equiv \sigma y, \tag{21}$$

where α is the R of Chandrasekhar,[6] ν is the kinematic coefficient of viscosity, and t is the time elapsed since the beginning of the decay. In terms of the function $f - g$ tabulated by Proudman, expressions (8), (10), and (19) take the form

$$\frac{y^2P_1(y) + 3P_2(y)}{\overline{u^2}^2} = -4\int_y^\infty (f - g)^2 \frac{1}{x}\, dx, \tag{22}$$

$$\frac{\sigma^2}{\overline{u^2}^2} P_1(y) = +\frac{24}{5}\int_y^\infty (f - g)^2 \frac{1}{x^3}\, dx + \frac{14}{5y^5}\int_0^y (f - g)^2 x^2\, dx, \tag{23}$$

and

$$y^2\sigma^2P_1(y) + P_2(y) \tag{24}$$

Using Proudman's tables for infinite Reynolds number, the integrals have been evaluated and the results are presented in table 1. From these values,

the correlations of expressions (19) and (20) have been plotted in figures 1 and 2.

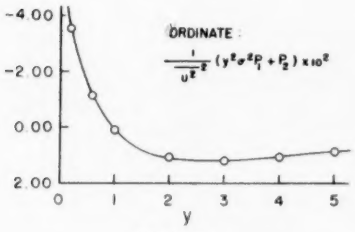

FIGURE 1

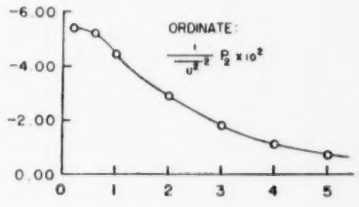

FIGURE 2

TABLE 1

y	0.2	0.6	1.0	2.0	3.0	4.0	5.0
$\frac{\sigma^2}{u^{22}} P_1 \times 10^2$	+457.	+112.	+45.3	+9.94	+3.32	+1.37	+0.64
$\frac{1}{u^{22}} (y^2 P_1 + 3P_2) \times 10^2$	−14.4	−11.6	−8.73	−4.72	−2.43	−1.26	−0.68
$\frac{1}{u^{22}} P_2 \times 10^2$	−5.40	−5.20	−4.43	−2.90	−1.81	−1.15	−0.77
$\frac{1}{u^{22}} (y^2\sigma^2 P_1 + P_2) \times 10^2$	−3.57	−1.17	+0.10	+1.07	+1.18	+1.04	+0.84

It should be possible to check these predicted correlations experimentally and thus obtain a better understanding concerning the underlying theory.

In conclusion I should like to express my gratitude to Professor Chandrasekhar who not only suggested the investigation but also has given generously of his time and patience in seeing it to its completion.

[1] Batchelor, G. K. (in press). I am indebted to Dr. Batchelor and to Professor Chandrasekhar for having made a manuscript copy of this paper available to me.

[2] Robertson, H. P., *Proc. Camb. Phil. Soc.*, **36,** 209 (1940).

[3] Chandrasekhar, S., *Proc. Roy. Soc. (London) A*, **204,** 435 (1951).

[4] Batchelor, G. K., *Proc. Camb. Phil. Soc.*, **43,** 533 (1947).

[5] Proudman, I. (in press). I am indebted to Drs. Proudman and Batchelor and to Professor Chandrasekhar for having made a manuscript copy of this paper available to me.

[6] Chandrasekhar, S., *Proc. Roy. Soc. (London) A*, **200,** 20 (1949).

ERRATA

In the article "On the Origin of the Solar System," these PROCEEDINGS, **37,** 1–14 (1951), the following corrections should be made:

Page 6, formula (4) for d read a.

Page 8, table 1, last column, for 1.0 read 1.10.

GERARD P. KUIPER

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